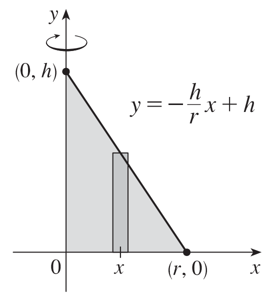
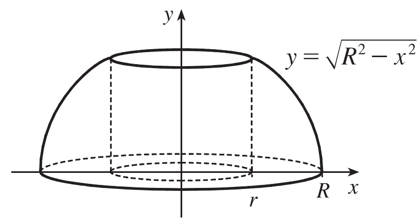
37. $V = 2\pi \int_0^r x\left(-\frac{h}{r}x + h\right) dx = 2\pi h \int_0^r \left(-\frac{x^2}{r} + x\right) dx$

$= 2\pi h\left[-\frac{x^3}{3r} + \frac{x^2}{2}\right]_0^r = 2\pi h \frac{r^2}{6} = \frac{\pi r^2 h}{3}$



38. By symmetry, the volume of a napkin ring obtained by drilling a hole of radius r through a sphere with radius R is twice the volume obtained by rotating the area above the x-axis and below the curve $y = \sqrt{R^2 - x^2}$ (the equation of the top half of the cross-section of the sphere), between $x = r$ and $x = R$, about the y-axis. This volume is equal to



$$2\int_{\text{inner radius}}^{\text{outer radius}} 2\pi r h\, dx = 2 \cdot 2\pi \int_r^R x\sqrt{R^2 - x^2}\, dx = 4\pi\left[-\tfrac{1}{3}\left(R^2 - x^2\right)^{3/2}\right]_r^R = \tfrac{4}{3}\pi(R^2 - r^2)^{3/2}$$

But by the Pythagorean Theorem, $R^2 - r^2 = \left(\frac{1}{2}h\right)^2$, so the volume of the napkin ring is $\frac{4}{3}\pi\left(\frac{1}{2}h\right)^3 = \frac{1}{6}\pi h^3$, which is independent of both R and r; that is, the amount of wood in a napkin ring of height h is the same regardless of the size of the sphere used. Note that most of this calculation has been done already, but with more difficulty, in Exercise 6.2.52.

Another solution: The height of the missing cap is the radius of the sphere minus half the height of the cut-out cylinder, that is, $R - \frac{1}{2}h$. Using Exercise 6.2.33,

$$V_{\text{napkin ring}} = V_{\text{sphere}} - V_{\text{cylinder}} - 2V_{\text{cap}} = \tfrac{4}{3}\pi R^3 - \pi r^2 h - 2 \cdot \tfrac{\pi}{3}\left(R - \tfrac{1}{2}h\right)^2\left[3R - \left(R - \tfrac{1}{2}h\right)\right] = \tfrac{1}{6}\pi h^3$$

6.4 Arc Length

1. $y = 2x - 5 \;\Rightarrow\; L = \int_{-1}^3 \sqrt{1 + (dy/dx)^2}\, dx = \int_{-1}^3 \sqrt{1 + (2)^2}\, dx = \sqrt{5}\,[3 - (-1)] = 4\sqrt{5}$.

The arc length can be calculated using the distance formula, since the curve is a line segment, so

$L = [\text{distance from } (-1, -7) \text{ to } (3, 1)] = \sqrt{[3 - (-1)]^2 + [1 - (-7)]^2} = \sqrt{80} = 4\sqrt{5}$

2. (a) $x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-\sin t)^2 + (\cos t)^2 = \sin^2 t + \cos^2 t = 1$. So by formula (1),

$L = \int_0^{2\pi} \sqrt{1}\, dt = \left[\,t\,\right]_0^{2\pi} = 2\pi$, as expected.

(b) $x = \sin 2t,\ y = \cos 2t,\ 0 \le t \le 2\pi$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (2\cos 2t)^2 + (-2\sin 2t)^2 = 4\cos^2 2t + 4\sin^2 2t = 4$.

$L = \int_0^{2\pi} \sqrt{4}\, dt = 2\left[\,t\,\right]_0^{2\pi} = 2(2\pi) = 4\pi$. The discrepancy results from the fact that the unit circle is traversed twice with this parametrization.

3. $y = \sin x \;\Rightarrow\; dy/dx = \cos x \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + \cos^2 x$. So $L = \int_0^\pi \sqrt{1 + \cos^2 x}\, dx \approx 3.8202$.

4. $x = y^2 - 2y \;\Rightarrow\; dx/dy = 2y - 2 \;\Rightarrow\; 1 + (dx/dy)^2 = 1 + (2y - 2)^2$. So $L = \int_0^2 \sqrt{1 + (2y - 2)^2}\, dy \approx 2.9579$.

5. $x = t + \cos t,\ \ y = t - \sin t,\ \ 0 \le t \le 2\pi.\quad dx/dt = 1 - \sin t$ and $dy/dt = 1 - \cos t$, so

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1-\sin t)^2 + (1-\cos t)^2 = (1 - 2\sin t + \sin^2 t) + (1 - 2\cos t + \cos^2 t) = 3 - 2\sin t - 2\cos t.$

Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^{2\pi} \sqrt{3 - 2\sin t - 2\cos t}\, dt \approx 10.0367.$

6. $x = t\cos t, y = t\sin t, 0 \le t \le 2\pi.\ \ \dfrac{dx}{dt} = t(-\sin t) + \cos t \cdot 1$ and $\dfrac{dy}{dt} = t \cdot \cos t + \sin t \cdot 1$, so

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= \left(t^2\sin^2 t - 2t\sin t\cos t + \cos^2 t\right) + \left(t^2\cos^2 t + 2t\sin t\cos t + \sin^2 t\right)\\ &= t^2\left(\sin^2 t + \cos^2 t\right) + \cos^2 t + \sin^2 t = t^2 + 1\end{aligned}$$

Thus, $L = \displaystyle\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{t^2+1}\, dt \approx 21.2563.$

7. $x = 1 + 3t^2,\ \ y = 4 + 2t^3,\ \ 0 \le t \le 1.$ $dx/dt = 6t$ and $dy/dt = 6t^2$, so $(dx/dt)^2 + (dy/dt)^2 = 36t^2 + 36t^4$

$$\begin{aligned}\text{Thus,}\quad L &= \int_0^1 \sqrt{36t^2 + 36t^4}\, dt = \int_0^1 6t\sqrt{1+t^2}\, dt = 6\int_1^2 \sqrt{u}\,\left(\tfrac{1}{2}du\right) \quad [u = 1 + t^2, du = 2t\,dt]\\ &= 3\left[\tfrac{2}{3}u^{3/2}\right]_1^2 = 2(2^{3/2} - 1) = 2\left(2\sqrt{2} - 1\right)\end{aligned}$$

8. $y^2 = 4(x+4)^3,\ y > 0 \quad\Rightarrow\quad y = 2(x+4)^{3/2} \quad\Rightarrow\quad dy/dx = 3(x+4)^{1/2} \quad\Rightarrow$

$1 + (dy/dx)^2 = 1 + 9(x+4) = 9x + 37.$ So

$$L = \int_0^2 \sqrt{9x+37}\, dx \quad \begin{bmatrix} u = 9x+37, \\ du = 9\,dx \end{bmatrix} \quad = \int_{37}^{55} u^{1/2}\left(\tfrac{1}{9}\,du\right) = \tfrac{1}{9}\cdot\tfrac{2}{3}\left[u^{3/2}\right]_{37}^{55} = \tfrac{2}{27}\left(55\sqrt{55} - 37\sqrt{37}\right).$$

9. $x = y^{3/2} \quad\Rightarrow\quad 1 + (dx/dy)^2 = 1 + \left(\tfrac{3}{2}y^{1/2}\right)^2 = 1 + \tfrac{9}{4}y.$

$$\begin{aligned}L &= \int_0^1 \sqrt{1 + \tfrac{9}{4}y}\, dy = \int_1^{13/4} \sqrt{u}\left(\tfrac{4}{9}\,du\right) \quad \left[u = 1 + \tfrac{9}{4}y, du = \tfrac{9}{4}\,dy\right]\\ &= \tfrac{4}{9}\cdot\tfrac{2}{3}\left[u^{3/2}\right]_1^{13/4} = \tfrac{8}{27}\left(\tfrac{13\sqrt{13}}{8} - 1\right) = \tfrac{13\sqrt{13} - 8}{27}.\end{aligned}$$

10. $y = \sqrt{x - x^2} + \sin^{-1}\left(\sqrt{x}\right) \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1-2x}{2\sqrt{x-x^2}} + \dfrac{1}{2\sqrt{x}\sqrt{1-x}} = \dfrac{2-2x}{2\sqrt{x}\sqrt{1-x}} = \sqrt{\dfrac{1-x}{x}} \quad\Rightarrow$

$1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{1-x}{x} = \dfrac{1}{x}.$ The curve has endpoints $(0,0)$ and $\left(1, \frac{\pi}{2}\right)$, so $L = \displaystyle\int_0^1 \sqrt{\frac{1}{x}}\, dx = \left[2\sqrt{x}\right]_0^1 = 2.$

11. $y = \dfrac{1}{4}x^2 - \dfrac{1}{2}\ln x \quad\Rightarrow\quad y' = \dfrac{1}{2}x - \dfrac{1}{2x} \quad\Rightarrow\quad 1 + (y')^2 = 1 + \left(\dfrac{1}{4}x^2 - \dfrac{1}{2} + \dfrac{1}{4x^2}\right) = \dfrac{1}{4}x^2 + \dfrac{1}{2} + \dfrac{1}{4x^2} = \left(\dfrac{1}{2}x + \dfrac{1}{2x}\right)^2.$

So

$$\begin{aligned}L &= \int_1^2 \sqrt{1 + (y')^2}\, dx = \int_1^2 \left|\frac{1}{2}x + \frac{1}{2x}\right| dx = \int_1^2 \left(\frac{1}{2}x + \frac{1}{2x}\right) dx\\ &= \left[\frac{1}{4}x^2 + \frac{1}{2}\ln|x|\right]_1^2 = \left(1 + \frac{1}{2}\ln 2\right) - \left(\frac{1}{4} + 0\right) = \frac{3}{4} + \frac{1}{2}\ln 2\end{aligned}$$

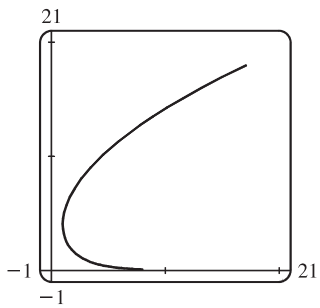
12. $x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta),\ 0 \le \theta \le \pi$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2[(-\sin\theta + \theta\cos\theta + \sin\theta)^2 + (\cos\theta + \theta\sin\theta - \cos\theta)^2]$$

$$= a^2\theta^2(\cos^2\theta + \sin^2\theta) = (a\theta)^2$$

$L = \int_0^\pi a\theta\,d\theta = a\left[\frac{1}{2}\theta^2\right]_0^\pi = \frac{1}{2}\pi^2 a$

13.



$x = e^t - t,\ y = 4e^{t/2},\ -8 \le t \le 3$

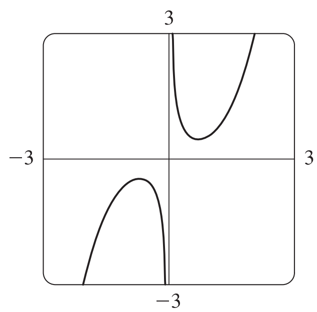
$$\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} - 2e^t + 1 + 4e^t$$

$$= e^{2t} + 2e^t + 1 = (e^t + 1)^2$$

$$L = \int_{-8}^{3} \sqrt{(e^t+1)^2}\,dt = \int_{-8}^{3} (e^t + 1)\,dt = \left[e^t + t\right]_{-8}^{3t}$$

$$= (e^3 + 3) - (e^{-8} - 8) = e^3 - e^{-8} + 11$$

14.



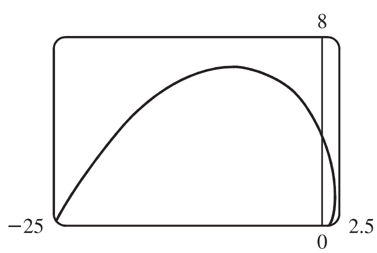
$$y = \frac{x^3}{3} + \frac{1}{4x} \quad\Rightarrow\quad y' = x^2 - \frac{1}{4x^2} \quad\Rightarrow$$

$$1 + (y')^2 = 1 + \left(x^4 - \frac{1}{2} + \frac{1}{16x^4}\right) = x^4 + \frac{1}{2} + \frac{1}{16x^4} = \left(x^2 + \frac{1}{4x^2}\right)^2.\ \text{So}$$

$$L = \int_1^2 \sqrt{1 + (y')^2}\,dx = \int_1^2 \left|x^2 + \frac{1}{4x^2}\right| dx = \int_1^2 \left(x^2 + \frac{1}{4x^2}\right) dx$$

$$= \left[\frac{1}{3}x^3 - \frac{1}{4x}\right]_1^2 = \left(\frac{8}{3} - \frac{1}{8}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{7}{3} + \frac{1}{8} = \frac{59}{24}$$

15.



$x = e^t\cos t,\ y = e^t\sin t,\ 0 \le t \le \pi.$

$$\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 = [e^t(\cos t - \sin t)]^2 + [e^t(\sin t + \cos t)]^2$$

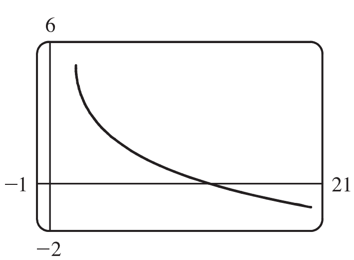
$$= (e^t)^2(\cos^2 t - 2\cos t\,\sin t + \sin^2 t)$$

$$+ (e^t)^2(\sin^2 t + 2\sin t\,\cos t + \cos^2 t$$

$$= e^{2t}(2\cos^2 t + 2\sin^2 t) = 2e^{2t}$$

Thus, $L = \int_0^\pi \sqrt{2e^{2t}}\,dt = \int_0^\pi \sqrt{2}\,e^t\,dt = \sqrt{2}\,\left[\,e^t\,\right]_0^\pi = \sqrt{2}\,(e^\pi - 1).$

16.



$x = e^t + e^{-t}, y = 5 - 2t, 0 \le t \le 3.$

$dx/dt = e^t - e^{-t}$ and $dy/dt = -2$, so

$\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 = e^{2t} - 2 + e^{-2t} + 4 = e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2$ and

$L = \int_0^3 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^3 = e^3 - e^{-3} - (1 - 1) = e^3 - e^{-3}.$

17. $y = xe^{-x} \quad \Rightarrow \quad dy/dx = e^{-x} - xe^{-x} = e^{-x}(1-x) \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + e^{-2x}(1-x)^2$. Let $f(x) = \sqrt{1 + (dy/dx)^2} = \sqrt{1 + e^{-2x}(1-x)^2}$. Then $L = \int_0^5 f(x)\,dx$. Since $n = 10$, $\Delta x = \frac{5-0}{10} = \frac{1}{2}$. Now

$$L \approx S_{10} = \tfrac{1/2}{3}\left[f(0) + 4f\left(\tfrac{1}{2}\right) + 2f(1) + 4f\left(\tfrac{3}{2}\right) + 2f(2) + 4f\left(\tfrac{5}{2}\right) + 2f(3) + 4f\left(\tfrac{7}{2}\right) + 2f(4) + 4f\left(\tfrac{9}{2}\right) + f(5)\right]$$
$$\approx 5.115840$$

The value of the integral produced by a calculator is 5.113568 (to six decimal places).

18. $x = y + \sqrt{y} \quad \Rightarrow \quad dx/dy = 1 + \dfrac{1}{2\sqrt{y}} \quad \Rightarrow \quad 1 + (dx/dy)^2 = 1 + \left(1 + \dfrac{1}{2\sqrt{y}}\right)^2 = 2 + \dfrac{1}{\sqrt{y}} + \dfrac{1}{4y}$.

Let $g(y) = \sqrt{1 + (dx/dy)^2}$. Then $L = \int_1^2 g(y)\,dy$. Since $n = 10$, $\Delta y = \frac{2-1}{10} = \frac{1}{10}$. Now

$$L \approx S_{10} = \tfrac{1/10}{3}[g(1) + 4g(1.1) + 2g(1.2) + 4g(1.3) + 2g(1.4)$$
$$+ 4g(1.5) + 2g(1.6) + 4g(1.7) + 2g(1.8) + 4g(1.9) + g(2)] \approx 1.732215,$$

which is the same value of the integral produced by a calculator to six decimal places.

19. $x = \sin t,\ y = t^2 \quad \Rightarrow \quad (dx/dt)^2 + (dy/dt)^2 = (\cos t)^2 + (2t)^2 = \cos^2 t + 4t^2 \quad \Rightarrow \quad L = \int_0^{2\pi} \sqrt{\cos^2 t + 4t^2}\,dt$.

Using Simpson's Rule with $n = 10$, $\Delta t = \dfrac{2\pi - 0}{10} = \dfrac{\pi}{5}$, and $f(t) = \sqrt{\cos^2 t + 4t^2}\,dt$, we get

$$L \approx S_{10} = \tfrac{2\pi - 0}{3(10)}\left[f(0) + 4f\left(\tfrac{\pi}{5}\right) + 2f\left(\tfrac{2\pi}{5}\right) + 4f\left(\tfrac{3\pi}{5}\right) + 2f\left(\tfrac{4\pi}{5}\right) + 4f(\pi) + 2f\left(\tfrac{6\pi}{5}\right)\right.$$
$$\left. + 4f\left(\tfrac{7\pi}{5}\right) + 2f\left(\tfrac{8\pi}{5}\right) + 4f\left(\tfrac{9\pi}{5}\right) + f(2\pi)\right]$$
$$\approx 40.056222$$

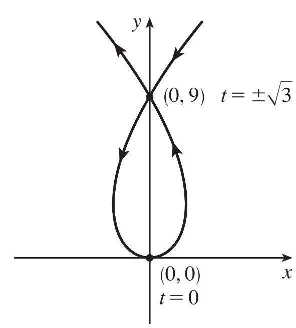
The value of the integral produced by a calculator is 40.051156 (to six decimal places).

20. $x = 3t - t^3,\ y = 3t^2$. $\quad dx/dt = 3 - 3t^2$ and $dy/dt = 6t$, so

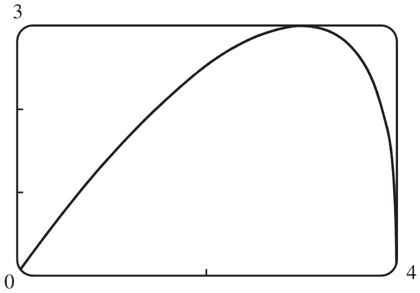
$$\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = (3 + 3t^2)^2$$

and the length of the loop is given by

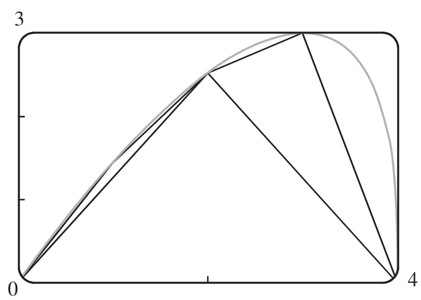
$$L = \int_{-\sqrt{3}}^{\sqrt{3}} (3 + 3t^2)\,dt = 2\int_0^{\sqrt{3}} (3 + 3t^2)\,dt = 2\left[3t + t^3\right]_0^{\sqrt{3}}$$
$$= 2\left(3\sqrt{3} + 3\sqrt{3}\right) = 12\sqrt{3}.$$



21. (a)



(b)



Let $f(x) = y = x\sqrt[3]{4-x}$. The polygon with one side is just the line segment joining the points $(0, f(0)) = (0,0)$ and $(4, f(4)) = (4,0)$, and its length $L_1 = 4$.

The polygon with two sides joins the points $(0,0)$, $(2, f(2)) = \left(2, 2\sqrt[3]{2}\right)$ and $(4,0)$. Its length

$$L_2 = \sqrt{(2-0)^2 + \left(2\sqrt[3]{2} - 0\right)^2} + \sqrt{(4-2)^2 + \left(0 - 2\sqrt[3]{2}\right)^2} = 2\sqrt{4 + 2^{8/3}} \approx 6.43$$

Similarly, the inscribed polygon with four sides joins the points $(0,0)$, $\left(1, \sqrt[3]{3}\right)$, $\left(2, 2\sqrt[3]{2}\right)$, $(3,3)$, and $(4,0)$, so its length

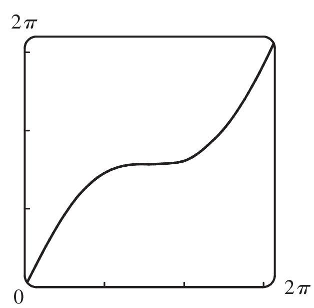
$$L_4 = \sqrt{1 + \left(\sqrt[3]{3}\right)^2} + \sqrt{1 + \left(2\sqrt[3]{2} - \sqrt[3]{3}\right)^2} + \sqrt{1 + \left(3 - 2\sqrt[3]{2}\right)^2} + \sqrt{1+9} \approx 7.50$$

(c) Using the arc length formula with $\dfrac{dy}{dx} = x\left[\frac{1}{3}(4-x)^{-2/3}(-1)\right] + \sqrt[3]{4-x} = \dfrac{12-4x}{3(4-x)^{2/3}}$, the length of the curve is

$$L = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^4 \sqrt{1 + \left[\frac{12-4x}{3(4-x)^{2/3}}\right]^2}\,dx.$$

(d) According to a calculator, the length of the curve is $L \approx 7.7988$. The actual value is larger than any of the approximations in part (b). This is always true, since any approximating straight line between two points on the curve is shorter than the length of the curve between the two points.

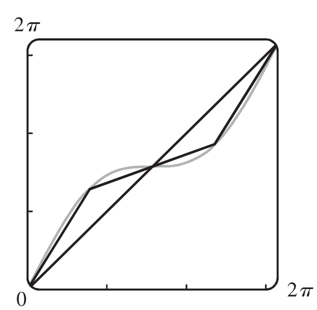
22. (a) Let $f(x) = y = x + \sin x$ with $0 \le x \le 2\pi$.



(b) The polygon with one side is just the line segment joining the points $(0, f(0)) = (0,0)$ and $(2\pi, f(2\pi)) = (2\pi, 2\pi)$, and its length is $\sqrt{(2\pi-0)^2 + (2\pi-0)^2} = 2\sqrt{2}\,\pi \approx 8.9$.

The polygon with two sides joins the points $(0,0)$, $(\pi, f(\pi)) = (\pi, \pi)$, and $(2\pi, 2\pi)$. Its length is

$$\sqrt{(\pi-0)^2 + (\pi-0)^2} + \sqrt{(2\pi-\pi)^2 + (2\pi-\pi)^2} = \sqrt{2}\,\pi + \sqrt{2}\,\pi$$
$$= 2\sqrt{2}\,\pi \approx 8.9$$



Note from the diagram that the two approximations are the same because the sides of the two-sided polygon are in fact on the same line, since $f(\pi) = \pi = \frac{1}{2}f(2\pi)$.

The four-sided polygon joins the points $(0,0)$, $\left(\frac{\pi}{2}, \frac{\pi}{2}+1\right)$, (π, π), $\left(\frac{3\pi}{2}, \frac{3\pi}{2} - 1\right)$, and $(2\pi, 2\pi)$, so its length is

$$\sqrt{\left(\tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2}+1\right)^2} + \sqrt{\left(\tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2}-1\right)^2} + \sqrt{\left(\tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2}-1\right)^2} + \sqrt{\left(\tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2}+1\right)^2} \approx 9.4$$

(c) Using the arc length formula with $dy/dx = 1 + \cos x$, the length of the curve is

$$L = \int_0^{2\pi} \sqrt{1 + (1 + \cos x)^2}\, dx = \int_0^{2\pi} \sqrt{2 + 2\cos x + \cos^2 x}\, dx$$

(d) The calculator approximates the integral as 9.5076. The actual length is larger than the approximations in part (b).

23. $x = t^3 \quad \Rightarrow \quad dx/dt = 3t^2$ and $y = t^4 \quad \Rightarrow \quad dy/dt = 4t^3$. So

$L = \int_0^1 \sqrt{9t^4 + 16t^6}\, dt = \int_0^1 \sqrt{t^4(9 + 16t^2)}\, dt = \int_0^1 t^2\sqrt{9 + 16t^2}\, dt.$

Now use Formula 22 from the table of integrals to evaluate L.

$$\begin{aligned} L &= \int_0^4 \left(\tfrac{1}{4}u\right)^2 \sqrt{a^2 + u^2}\, \left(\tfrac{1}{4}du\right) \qquad [a = 3,\, u = 4t,\, du = 4\,dt] \\ &= \tfrac{1}{64}\int_0^4 u^2\sqrt{a^2 + u^2}\, du = \tfrac{1}{64}\left[\tfrac{u}{8}\left(9 + 2u^2\right)\sqrt{9 + u^2} - \tfrac{81}{8}\ln\left(u + \sqrt{9 + u^2}\,\right)\right]_0^4 \\ &= \tfrac{1}{64}\left\{\left[\tfrac{1}{2}\cdot 41\cdot 5 - \tfrac{81}{8}\ln(4 + 5)\right] - \left[0 - \tfrac{81}{8}\ln 3\right]\right\} \\ &= \tfrac{1}{64}\left[\tfrac{205}{2} - \tfrac{81}{8}(2\ln 3) + \tfrac{81}{8}\ln 3\right] \qquad \left[\ln 9 = \ln 3^2 = 2\ln 3\right] \\ &= \tfrac{1}{64}\left(\tfrac{205}{2} - \tfrac{81}{8}\ln 3\right) = \tfrac{205}{128} - \tfrac{81}{512}\ln 3 \approx 1.428. \end{aligned}$$

24. $y^2 = 4x,\, x = \frac{1}{4}y^2 \quad \Rightarrow \quad dx/dy = \frac{1}{2}y \quad \Rightarrow \quad 1 + (dx/dy)^2 = 1 + \frac{1}{4}y^2$. So

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + \tfrac{1}{4}y^2}\, dy = \int_0^1 \sqrt{1 + u^2}\cdot 2\, du \qquad \left[u = \tfrac{1}{2}y,\, dy = 2\, du\right] \\ &\overset{21}{=} \left[u\sqrt{1 + u^2} + \ln\left|u + \sqrt{1 + u^2}\,\right|\right]_0^1 = \sqrt{2} + \ln\left(1 + \sqrt{2}\,\right) \end{aligned}$$

25. $y = \ln(\cos x) \quad \Rightarrow \quad y' = \dfrac{1}{\cos x}(-\sin x) = -\tan x \quad \Rightarrow \quad 1 + (y')^2 = 1 + \tan^2 x = \sec^2 x.$

So $L = \int_0^{\pi/4} \sec x\, dx \overset{14}{=} \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4} = \ln\left(\sqrt{2} + 1\right) - \ln(1 + 0) = \ln\left(\sqrt{2} + 1\right) \approx 0.881.$

26. $y = \ln x \quad \Rightarrow \quad y' = \dfrac{1}{x} \quad \Rightarrow \quad 1 + (y')^2 = 1 + \dfrac{1}{x^2} = \dfrac{x^2 + 1}{x^2}$. So

$$\begin{aligned} L &= \int_1^{\sqrt{3}} \sqrt{\frac{x^2 + 1}{x^2}}\, dx = \int_1^{\sqrt{3}} \frac{\sqrt{x^2 + 1}}{x}\, dx \overset{23}{=} \left[\sqrt{x^2 + 1} - \ln\left|\frac{1 + \sqrt{x^2 + 1}}{x}\right|\right]_1^{\sqrt{3}} \\ &= \left(2 - \ln\sqrt{3}\,\right) - \left(\sqrt{2} - \ln\left(1 + \sqrt{2}\,\right)\right) = 2 - \sqrt{2} + \ln\left(1 + \sqrt{2}\,\right) - \ln\sqrt{3} \end{aligned}$$

27. The prey hits the ground when $y = 0 \quad \Leftrightarrow \quad 180 - \frac{1}{45}x^2 = 0 \quad \Leftrightarrow \quad x^2 = 45\cdot 180 \quad \Rightarrow \quad x = \sqrt{8100} = 90,$

since x must be positive. $y' = -\frac{2}{45}x \quad \Rightarrow \quad 1 + (y')^2 = 1 + \frac{4}{45^2}x^2$, so the distance traveled by the prey is

$$\begin{aligned} L &= \int_0^{90} \sqrt{1 + \frac{4}{45^2}x^2}\, dx = \int_0^4 \sqrt{1 + u^2}\left(\tfrac{45}{2}\, du\right) \qquad \begin{bmatrix} u = \frac{2}{45}x, \\ du = \frac{2}{45}\, dx \end{bmatrix} \\ &\overset{21}{=} \tfrac{45}{2}\left[\tfrac{1}{2}u\sqrt{1 + u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1 + u^2}\,\right)\right]_0^4 = \tfrac{45}{2}\left[2\sqrt{17} + \tfrac{1}{2}\ln\left(4 + \sqrt{17}\,\right)\right] = 45\sqrt{17} + \tfrac{45}{4}\ln\left(4 + \sqrt{17}\,\right) \approx 209.1 \text{ m} \end{aligned}$$

28. $y = 150 - \frac{1}{40}(x-50)^2 \;\Rightarrow\; y' = -\frac{1}{20}(x-50) \;\Rightarrow\; 1+(y')^2 = 1 + \frac{1}{20^2}(x-50)^2$, so the distance traveled by the kite is

$$L = \int_0^{80} \sqrt{1 + \frac{1}{20^2}(x-50)^2}\,dx = \int_{-5/2}^{3/2} \sqrt{1+u^2}\,(20\,du) \qquad \left[\begin{aligned} u &= \tfrac{1}{20}(x-50), \\ du &= \tfrac{1}{20}\,dx \end{aligned}\right]$$

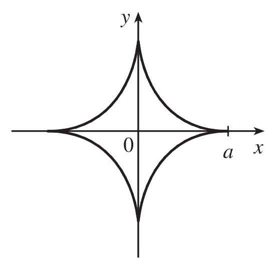
$$\overset{21}{=} 20\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\!\left(u+\sqrt{1+u^2}\right)\right]_{-5/2}^{3/2} = 10\left[\tfrac{3}{2}\sqrt{\tfrac{13}{4}} + \ln\!\left(\tfrac{3}{2}+\sqrt{\tfrac{13}{4}}\right) + \tfrac{5}{2}\sqrt{\tfrac{29}{4}} - \ln\!\left(-\tfrac{5}{2}+\sqrt{\tfrac{29}{4}}\right)\right]$$

$$= \tfrac{15}{2}\sqrt{13} + \tfrac{25}{2}\sqrt{29} + 10\ln\!\left(\frac{3+\sqrt{13}}{-5+\sqrt{29}}\right) \approx 122.8 \text{ ft}$$

29. The sine wave has amplitude 1 and period 14, since it goes through two periods in a distance of 28 in., so its equation is $y = 1\sin\!\left(\frac{2\pi}{14}x\right) = \sin\!\left(\frac{\pi}{7}x\right)$. The width w of the flat metal sheet needed to make the panel is the arc length of the sine curve from $x = 0$ to $x = 28$. We set up the integral to evaluate w using the arc length formula with $\frac{dy}{dx} = \frac{\pi}{7}\cos\!\left(\frac{\pi}{7}x\right)$:

$L = \int_0^{28} \sqrt{1 + \left[\frac{\pi}{7}\cos\!\left(\frac{\pi}{7}x\right)\right]^2}\,dx = 2\int_0^{14} \sqrt{1 + \left[\frac{\pi}{7}\cos\!\left(\frac{\pi}{7}x\right)\right]^2}\,dx$. This integral would be very difficult to evaluate exactly, so we use a CAS, and find that $L \approx 29.36$ inches.

30. $x = a\cos^3\theta,\ y = a\sin^3\theta$.



$$\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 = (-3a\cos^2\theta\,\sin\theta)^2 + (3a\sin^2\theta\,\cos\theta)^2$$
$$= 9a^2\cos^4\theta\,\sin^2\theta + 9a^2\sin^4\theta\,\cos^2\theta$$
$$= 9a^2\sin^2\theta\,\cos^2\theta(\cos^2\theta + \sin^2\theta) = 9a^2\sin^2\theta\,\cos^2\theta.$$

The graph has four-fold symmetry and the curve in the first quadrant corresponds to $0 \le \theta \le \pi/2$. Thus,

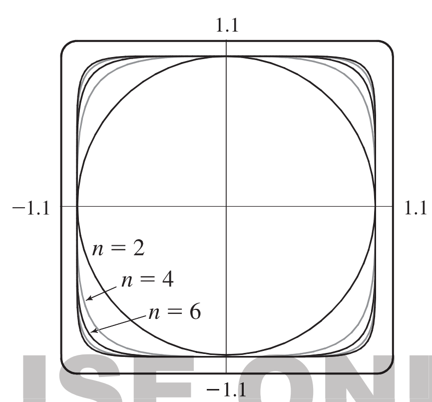
$$L = 4\int_0^{\pi/2} 3a\sin\theta\,\cos\theta\,d\theta \qquad [\text{since } a > 0 \text{ and } \sin\theta \text{ and } \cos\theta \text{ are positive for } 0 \le \theta \le \pi/2]$$
$$= 12a\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} = 12a\left(\tfrac{1}{2} - 0\right) = 6a$$

31. $x = a\sin\theta,\ y = b\cos\theta,\ 0 \le \theta \le 2\pi$.

$$\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 = (a\cos\theta)^2 + (-b\sin\theta)^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2(1-\sin^2\theta) + b^2\sin^2\theta$$
$$= a^2 - (a^2 - b^2)\sin^2\theta = a^2 - c^2\sin^2\theta = a^2\left(1 - \frac{c^2}{a^2}\sin^2\theta\right) = a^2(1 - e^2\sin^2\theta)$$

So $L = 4\int_0^{\pi/2} \sqrt{a^2\left(1 - e^2\sin^2\theta\right)}\,d\theta$ [by symmetry] $= 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\,d\theta$.

32. By symmetry, the length of the curve in each quadrant is the same, so we'll find the length in the first quadrant and multiply by 4.



$x^{2k} + y^{2k} = 1 \;\Rightarrow\; y^{2k} = 1 - x^{2k} \;\Rightarrow\; y = (1 - x^{2k})^{1/(2k)}$

(in the first quadrant), so we use the arc length formula with

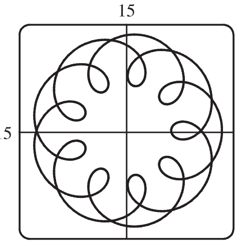
$$\frac{dy}{dx} = \frac{1}{2k}(1 - x^{2k})^{1/(2k)-1}(-2kx^{2k-1}) = -x^{2k-1}(1 - x^{2k})^{1/(2k)-1}$$

The total length is therefore

$$L_{2k} = 4\int_0^1 \sqrt{1 + [-x^{2k-1}(1 - x^{2k})^{1/(2k)-1}]^2}\,dx = 4\int_0^1 \sqrt{1 + x^{2(2k-1)}(1 - x^{2k})^{1/k-2}}\,dx$$

Now from the graph, we see that as k increases, the "corners" of these fat circles get closer to the points $(\pm1, \pm1)$ and $(\pm1, \mp1)$, and the "edges" of the fat circles approach the lines joining these four points. It seems plausible that as $k \to \infty$, the total length of the fat circle with $n = 2k$ will approach the length of the perimeter of the square with sides of length 2. This is supported by taking the limit as $k \to \infty$ of the equation of the fat circle in the first quadrant: $\lim\limits_{k\to\infty} (1 - x^{2k})^{1/(2k)} = 1$ for $0 \le x < 1$. So we guess that $\lim\limits_{k\to\infty} L_{2k} = 4 \cdot 2 = 8$.

33. (a) $x = 11\cos t - 4\cos(11t/2),\ y = 11\sin t - 4\sin(11t/2)$.

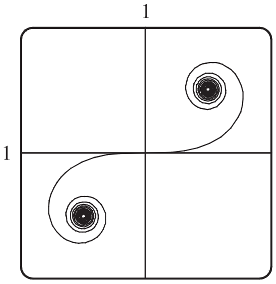


Notice that $0 \le t \le 2\pi$ does not give the complete curve because $x(0) \ne x(2\pi)$. In fact, we must take $t \in [0, 4\pi]$ in order to obtain the complete curve, since the first term in each of the parametric equations has period 2π and the second has period $\frac{2\pi}{11/2} = \frac{4\pi}{11}$, and the least common integer multiple of these two numbers is 4π.

(b) We use the CAS to find the derivatives dx/dt and dy/dt, and then use Formula 1 to find the arc length. Recent versions of Maple express the integral $\int_0^{4\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$ as $88E\big(2\sqrt{2}\,i\big)$, where $E(x)$ is the elliptic integral $\displaystyle\int_0^1 \frac{\sqrt{1 - x^2t^2}}{\sqrt{1 - t^2}}\,dt$ and i is the imaginary number $\sqrt{-1}$.

Some earlier versions of Maple (as well as Mathematica) cannot do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2),t=0..4*Pi));` to estimate the length, and find that the arc length is approximately 294.03. Derive's `Para_arc_length` function in the utility file `Int_apps` simplifies the integral to $11\int_0^{4\pi} \sqrt{-4\cos t\,\cos\!\left(\frac{11t}{2}\right) - 4\sin t\,\sin\!\left(\frac{11t}{2}\right) + 5}\,dt$.

34. (a) It appears that as $t \to \infty$, $(x, y) \to \left(\frac{1}{2}, \frac{1}{2}\right)$, and as $t \to -\infty$, $(x, y) \to \left(-\frac{1}{2}, -\frac{1}{2}\right)$.



(b) By the Fundamental Theorem of Calculus, $dx/dt = \cos\!\left(\frac{\pi}{2}t^2\right)$ and $dy/dt = \sin\!\left(\frac{\pi}{2}t^2\right)$, so by Formula 4, the length of the curve from the origin to the point with parameter value t is

$$L = \int_0^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\,du = \int_0^t \sqrt{\cos^2\!\left(\frac{\pi}{2}u^2\right) + \sin^2\!\left(\frac{\pi}{2}u^2\right)}\,du$$

$$= \int_0^t 1\,du = t \qquad [\text{or } -t \text{ if } t < 0]$$

We have used u as the dummy variable so as not to confuse it with the upper limit of integration.

DISCOVERY PROJECT Arc Length Contest

For advice on how to run the contest and a list of student entries, see the article "Arc Length Contest" by Larry Riddle in *The College Mathematics Journal*, Volume 29, No. 4, September 1998, pages 314–320.

6.5 Average Value of a Function

1. $f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-0}\int_0^4 (4x - x^2)\,dx = \frac{1}{4}\left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{1}{4}\left[\left(32 - \frac{64}{3}\right) - 0\right] = \frac{1}{4}\left(\frac{32}{3}\right) = \frac{8}{3}$

2. $f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi} \sin 4x\,dx = 0$ [by Theorem 5.5.6(b)]

3. $g_{\text{ave}} = \frac{1}{b-a}\int_a^b g(x)\,dx = \frac{1}{8-1}\int_1^8 \sqrt[3]{x}\,dx = \frac{1}{7}\left[\frac{3}{4}x^{4/3}\right]_1^8 = \frac{3}{28}(16-1) = \frac{45}{28}$

4. $f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} \sec^2(\theta/2)\,d\theta = \frac{2}{\pi}\left[2\tan(\theta/2)\right]_0^{\pi/2} = \frac{2}{\pi}\,[2(1) - 0] = \frac{4}{\pi}$

5. $h_{\text{ave}} = \frac{1}{\pi - 0}\int_0^{\pi} \cos^4 x\,\sin x\,dx = \frac{1}{\pi}\int_1^{-1} u^4(-du)$ $[u = \cos x,\, du = -\sin x\,dx]$

$= \frac{1}{\pi}\int_{-1}^{1} u^4\,du = \frac{1}{\pi}\cdot 2\int_0^1 u^4\,du$ [by Theorem 5.5.6(a)] $= \frac{2}{\pi}\left[\frac{1}{5}u^5\right]_0^1 = \frac{2}{5\pi}$

6. $h_{\text{ave}} = \frac{1}{b-a}\int_a^b h(u)\,du = \frac{1}{1-(-1)}\int_{-1}^{1}(3-2u)^{-1}\,du = \frac{1}{2}\int_{-1}^{1}\frac{1}{3-2u}\,du = \frac{1}{2}\int_5^1 \frac{1}{y}\left(-\frac{1}{2}\,dy\right)$ $[y = 3-2u,\, dy = -2\,du]$

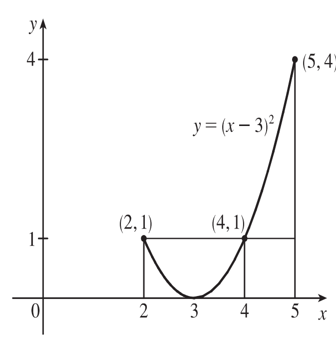
$= -\frac{1}{4}\Big[\ln|y|\Big]_5^1 = -\frac{1}{4}(\ln 1 - \ln 5) = \frac{1}{4}\ln 5$

7. (a) $f_{\text{ave}} = \dfrac{1}{5-2}\displaystyle\int_2^5 (x-3)^2\,dx = \frac{1}{3}\left[\frac{1}{3}(x-3)^3\right]_2^5$

$= \frac{1}{9}\left[2^3 - (-1)^3\right] = \frac{1}{9}(8+1) = 1$

(b) $f(c) = f_{\text{ave}} \Leftrightarrow (c-3)^2 = 1 \Leftrightarrow$
$c - 3 = \pm 1 \Leftrightarrow c = 2$ or 4

(c)



8. (a) $f_{\text{ave}} = \frac{1}{3-1}\int_1^3 \ln x\,dx = \frac{1}{2}[x\ln x - x]_1^3$ [by parts]

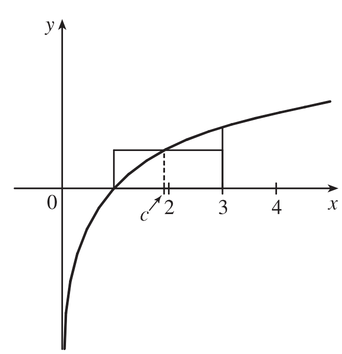
$= \frac{1}{2}[(3\ln 3 - 3) - (\ln 1 - 1)]$

$= \frac{1}{2}(3\ln 3 - 2) = \frac{3}{2}\ln 3 - 1$

(b) $f_{\text{ave}} = f(c) \Leftrightarrow \frac{3}{2}\ln 3 - 1 = \ln c \Leftrightarrow$

$c = e^{(3/2)\ln 3 - 1}$ or $c = 3\sqrt{3}/e \approx 1.91$

(c)



9. (a) $f_{\text{ave}} = \dfrac{1}{\pi - 0}\displaystyle\int_0^{\pi} (2\sin x - \sin 2x)\,dx$

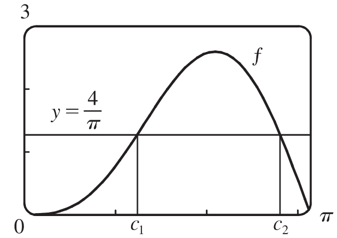
$= \frac{1}{\pi}\left[-2\cos x + \frac{1}{2}\cos 2x\right]_0^{\pi}$

$= \frac{1}{\pi}\left[\left(2 + \frac{1}{2}\right) - \left(-2 + \frac{1}{2}\right)\right] = \frac{4}{\pi}$

(b) $f(c) = f_{\text{ave}} \quad\Leftrightarrow\quad 2\sin c - \sin 2c = \frac{4}{\pi} \quad\Leftrightarrow$

$c_1 \approx 1.238$ or $c_2 \approx 2.808$

(c)



10. (a) $f_{\text{ave}} = \dfrac{1}{2 - 0}\displaystyle\int_0^{2} \frac{2x}{(1+x^2)^2}\,dx$

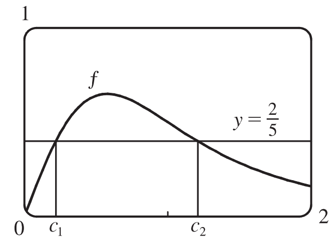
$= \dfrac{1}{2}\displaystyle\int_1^{5} \frac{1}{u^2}\,du \qquad [u = 1 + x^2,\, du = 2x\,dx]$

$= \dfrac{1}{2}\left[-\dfrac{1}{u}\right]_1^5 = -\dfrac{1}{2}\left(\dfrac{1}{5} - 1\right) = \dfrac{2}{5}$

(b) $f(c) = f_{\text{ave}} \quad\Leftrightarrow\quad \dfrac{2c}{(1+c^2)^2} = \dfrac{2}{5} \quad\Leftrightarrow\quad 5c = (1+c^2)^2 \quad\Leftrightarrow$

$c_1 \approx 0.220$ or $c_2 \approx 1.207$

(c)



11. f is continuous on $[1, 3]$, so by the Mean Value Theorem for Integrals there exists a number c in $[1, 3]$ such that $\int_1^3 f(x)\,dx = f(c)(3-1) \quad\Rightarrow\quad 8 = 2f(c)$; that is, there is a number c such that $f(c) = \frac{8}{2} = 4$.

12. The requirement is that $\dfrac{1}{b-0}\displaystyle\int_0^b f(x)\,dx = 3$. The LHS of this equation is equal to

$\dfrac{1}{b}\displaystyle\int_0^b (2 + 6x - 3x^2)\,dx = \dfrac{1}{b}\left[2x + 3x^2 - x^3\right]_0^b = 2 + 3b - b^2$, so we solve the equation $2 + 3b - b^2 = 3 \quad\Leftrightarrow$

$b^2 - 3b + 1 = 0 \quad\Leftrightarrow\quad b = \dfrac{3 \pm \sqrt{(-3)^2 - 4\cdot 1\cdot 1}}{2\cdot 1} = \dfrac{3 \pm \sqrt{5}}{2}$. Both roots are valid since they are positive.

13. $f_{\text{ave}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx \approx \dfrac{1}{50-20}S_6$

$= \frac{1}{30} \cdot \frac{50-20}{6\cdot 3}[f(20) + 4f(25) + 2f(30) + 4f(35) + 2f(40) + 4f(45) + f(50)]$

$= \frac{1}{18}[42 + 4(38) + 2(31) + 4(29) + 2(35) + 4(48) + 60] = \frac{1}{18}(694) = \frac{347}{9} \approx 38.6$

14. (a) $v_{\text{ave}} = \frac{1}{12-0}\int_0^{12} v(t)\,dt = \frac{1}{12}I$. Use the Midpoint Rule with $n = 3$ and $\Delta t = \frac{12-0}{3} = 4$ to estimate I.

$I \approx M_3 = 4[v(2) + v(6) + v(10)] = 4[21 + 50 + 66] = 4(137) = 548$. Thus, $v_{\text{ave}} \approx \frac{1}{12}(548) = 45\frac{2}{3}$ km/h.

(b) Estimating from the graph, $v(t) = 45\frac{2}{3}$ when $t \approx 5.2$ s.

15. Let $t = 0$ and $t = 12$ correspond to 9 AM and 9 PM, respectively.

$$T_{\text{ave}} = \frac{1}{12-0}\int_0^{12}\left[50 + 14\sin\tfrac{1}{12}\pi t\right]dt = \frac{1}{12}\left[50t - 14\cdot\tfrac{12}{\pi}\cos\tfrac{1}{12}\pi t\right]_0^{12}$$
$$= \frac{1}{12}\left[50\cdot 12 + 14\cdot\tfrac{12}{\pi} + 14\cdot\tfrac{12}{\pi}\right] = \left(50 + \tfrac{28}{\pi}\right)\,^\circ\text{F} \approx 59\,^\circ\text{F}$$

16. $T_{\text{ave}} = \dfrac{1}{30-0}\displaystyle\int_0^{30} (20 + 75e^{-t/50})\,dt = \dfrac{1}{30}\Big[20t - 50 \cdot 75e^{-t/50}\Big]_0^{30} = \dfrac{1}{30}[(600 - 3750e^{-3/5}) - (-3750)]$

$= \frac{1}{30}(4350 - 3750e^{-3/5}) = 145 - 125e^{-3/5} \approx 76.4\,^\circ\text{C}$

17. $\rho_{\text{ave}} = \dfrac{1}{8}\displaystyle\int_0^8 \dfrac{12}{\sqrt{x+1}}\,dx = \dfrac{3}{2}\displaystyle\int_0^8 (x+1)^{-1/2}\,dx = \big[3\sqrt{x+1}\,\big]_0^8 = 9 - 3 = 6\text{ kg/m}$

18. $s = \frac{1}{2}gt^2 \;\Rightarrow\; t = \sqrt{2s/g}$ [since $t \geq 0$]. Now $v = ds/dt = gt = g\sqrt{2s/g} = \sqrt{2gs} \;\Rightarrow\; v^2 = 2gs \;\Rightarrow\; s = \dfrac{v^2}{2g}$.

We see that v can be regarded as a function of t or of s: $v = F(t) = gt$ and $v = G(s) = \sqrt{2gs}$. Note that $v_T = F(T) = gT$.

Displacement can be viewed as a function of t: $s = s(t) = \frac{1}{2}gt^2$; also $s(t) = \dfrac{v^2}{2g} = \dfrac{[F(t)]^2}{2g}$. When $t = T$, these two formulas for $s(t)$ imply that

$$\sqrt{2gs(T)} = F(T) = v_T = gT = 2\big(\tfrac{1}{2}gT^2\big)/T = 2s(T)/T \qquad (\star)$$

The average of the velocities with respect to time t during the interval $[0, T]$ is

$$v_{t\text{-ave}} = F_{\text{ave}} = \frac{1}{T-0}\int_0^T F(t)\,dt = \frac{1}{T}\,[s(T) - s(0)] \quad \text{[by FTC]} \quad = \frac{s(T)}{T} \quad \text{[since } s(0) = 0] \quad = \frac{1}{2}v_T \quad \text{[by } (\star)]$$

But the average of the velocities with respect to displacement s during the corresponding displacement interval $[s(0), s(T)] = [0, s(T)]$ is

$$v_{s\text{-ave}} = G_{\text{ave}} = \frac{1}{s(T)-0}\int_0^{s(T)} G(s)\,ds = \frac{1}{s(T)}\int_0^{s(T)} \sqrt{2gs}\,ds = \frac{\sqrt{2g}}{s(T)}\int_0^{s(T)} s^{1/2}\,ds$$

$$= \frac{\sqrt{2g}}{s(T)} \cdot \frac{2}{3}\Big[s^{3/2}\Big]_0^{s(T)} = \frac{2}{3} \cdot \frac{\sqrt{2g}}{s(T)} \cdot \Big[s(T)\Big]^{3/2} = \frac{2}{3}\sqrt{2gs(T)} = \frac{2}{3}v_T \quad \text{[by } (\star)]$$

19. $V_{\text{ave}} = \frac{1}{5}\int_0^5 V(t)\,dt = \frac{1}{5}\int_0^5 \frac{5}{4\pi}\left[1 - \cos\!\left(\frac{2}{5}\pi t\right)\right] dt = \frac{1}{4\pi}\int_0^5 \left[1 - \cos\!\left(\frac{2}{5}\pi t\right)\right] dt$

$= \frac{1}{4\pi}\left[t - \frac{5}{2\pi}\sin\!\left(\frac{2}{5}\pi t\right)\right]_0^5 = \frac{1}{4\pi}\left[(5 - 0) - 0\right] = \frac{5}{4\pi} \approx 0.4\text{ L}$

20. $v_{\text{ave}} = \dfrac{1}{R-0}\displaystyle\int_0^R v(r)\,dr = \dfrac{1}{R}\displaystyle\int_0^R \dfrac{P}{4\eta l}(R^2 - r^2)\,dr = \dfrac{P}{4\eta lR}\left[R^2 r - \tfrac{1}{3}r^3\right]_0^R = \dfrac{P}{4\eta lR}\left(\tfrac{2}{3}\right)R^3 = \dfrac{PR^2}{6\eta l}$.

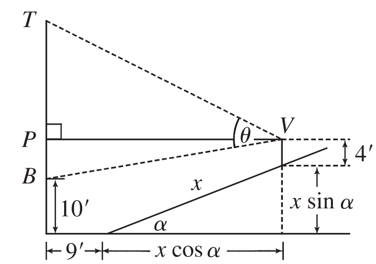
Since $v(r)$ is decreasing on $(0, R]$, $v_{\max} = v(0) = \dfrac{PR^2}{4\eta l}$. Thus, $v_{\text{ave}} = \frac{2}{3}v_{\max}$.

21. Let $F(x) = \int_a^x f(t)\,dt$ for x in $[a, b]$. Then F is continuous on $[a, b]$ and differentiable on (a, b), so by the Mean Value Theorem there is a number c in (a, b) such that $F(b) - F(a) = F'(c)(b - a)$. But $F'(x) = f(x)$ by the Fundamental Theorem of Calculus. Therefore, $\int_a^b f(t)\,dt - 0 = f(c)(b - a)$.

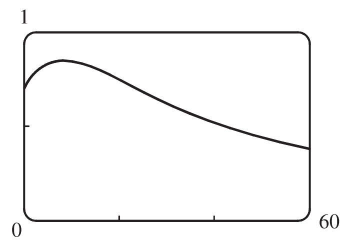
22. $f_{\text{ave}}\,[a, b] = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx = \dfrac{1}{b-a}\displaystyle\int_a^c f(x)\,dx + \dfrac{1}{b-a}\displaystyle\int_c^b f(x)\,dx$

$= \dfrac{c-a}{b-a}\left[\dfrac{1}{c-a}\displaystyle\int_a^c f(x)\,dx\right] + \dfrac{b-c}{b-a}\left[\dfrac{1}{b-c}\displaystyle\int_c^b f(x)\,dx\right] = \dfrac{c-a}{b-a}\,f_{\text{ave}}\,[a, c] + \dfrac{b-c}{b-a}\,f_{\text{ave}}\,[c, b]$

APPLIED PROJECT Where To Sit at the Movies



1. $|VP| = 9 + x\cos\alpha$, $|PT| = 35 - (4 + x\sin\alpha) = 31 - x\sin\alpha$, and $|PB| = (4 + x\sin\alpha) - 10 = x\sin\alpha - 6$. So using the Pythagorean Theorem, we have $|VT| = \sqrt{|VP|^2 + |PT|^2} = \sqrt{(9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2} = a$, and $|VB| = \sqrt{|VP|^2 + |PB|^2} = \sqrt{(9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2} = b$. Using the Law of Cosines on $\triangle VBT$, we get $25^2 = a^2 + b^2 - 2ab\cos\theta \quad\Leftrightarrow$ $\cos\theta = \dfrac{a^2 + b^2 - 625}{2ab} \quad\Leftrightarrow\quad \theta = \arccos\left(\dfrac{a^2 + b^2 - 625}{2ab}\right)$, as required.



2. From the graph of θ, it appears that the value of x which maximizes θ is $x \approx 8.25$ ft. Assuming that the first row is at $x = 0$, the row closest to this value of x is the fourth row, at $x = 9$ ft, and from the graph, the viewing angle in this row seems to be about 0.85 radians, or about 49°.

3. With a CAS, we type in the definition of θ, substitute in the proper values of a and b in terms of x and $\alpha = 20^\circ = \frac{\pi}{9}$ radians, and then use the differentiation command to find the derivative. We use a numerical rootfinder and find that the root of the equation $d\theta/dx = 0$ is $x \approx 8.253062$, as approximated in Problem 2.

4. From the graph in Problem 2, it seems that the average value of the function on the interval $[0, 60]$ is about 0.6. We can use a CAS to approximate $\frac{1}{60}\int_0^{60} \theta(x)\,dx \approx 0.625 \approx 36^\circ$. (The calculation is much faster if we reduce the number of digits of accuracy required.) The minimum value is $\theta(60) \approx 0.38$ and, from Problem 2, the maximum value is about 0.85.

6.6 Applications to Physics and Engineering

1. $W = \displaystyle\int_a^b f(x)\,dx = \int_0^9 \frac{10}{(1+x)^2}\,dx = 10\int_1^{10} \frac{1}{u^2}\,du \quad [u = 1 + x,\ du = dx] \quad = 10\left[-\frac{1}{u}\right]_1^{10} = 10\left(-\tfrac{1}{10} + 1\right) = 9$ ft-lb

2. $W = \int_1^2 \cos\left(\frac{1}{3}\pi x\right)dx = \frac{3}{\pi}\left[\sin\left(\frac{1}{3}\pi x\right)\right]_1^2 = \frac{3}{\pi}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = 0\text{ N·m} = 0\text{ J}$.

Interpretation: From $x = 1$ to $x = \frac{3}{2}$, the force does work equal to $\int_1^{3/2} \cos\left(\frac{1}{3}\pi x\right)dx = \frac{3}{\pi}\left(1 - \frac{\sqrt{3}}{2}\right)$ J in accelerating the particle and increasing its kinetic energy. From $x = \frac{3}{2}$ to $x = 2$, the force opposes the motion of the particle, decreasing its kinetic energy. This is negative work, equal in magnitude but opposite in sign to the work done from $x = 1$ to $x = \frac{3}{2}$.

3. The force function is given by $F(x)$ (in newtons) and the work (in joules) is the area under the curve, given by $\int_0^8 F(x)\,dx = \int_0^4 F(x)\,dx + \int_4^8 F(x)\,dx = \frac{1}{2}(4)(30) + (4)(30) = 180$ J.

4. Work $= \int_0^{18} f(x)\,dx \approx S_6 = \frac{18-0}{6\cdot 3}\left[f(0) + 4f(3) + 2f(6) + 4f(9) + 2f(12) + 4f(15) + f(18)\right]$

$$= 1\cdot[9.8 + 4(9.1) + 2(8.5) + 4(8.0) + 2(7.7) + 4(7.5) + 7.4] = 148 \text{ joules}$$

5. According to Hooke's Law, the force required to maintain a spring stretched x units beyond its natural length is proportional to x, that is, $f(x) = kx$. Here, the amount stretched is 4 in. $= \frac{1}{3}$ ft and the force is 10 lb. Thus, $10 = k\left(\frac{1}{3}\right) \Rightarrow k = 30$ lb/ft, and $f(x) = 30x$. The work done in stretching the spring from its natural length to 6 in. $= \frac{1}{2}$ ft beyond its natural length is $W = \int_0^{1/2} 30x\,dx = \left[15x^2\right]_0^{1/2} = \frac{15}{4}$ ft-lb.

6. According to Hooke's Law, the force required to maintain a spring stretched x units beyond its natural length is proportional to x, that is, $f(x) = kx$. Here, the amount stretched is $30 - 20 = 10\,\text{cm} = 0.1\,\text{m}$ and the force is $25\,\text{N}$. Thus, $25 = k(0.1) \Rightarrow k = 250\,\text{N/m}$, and $f(x) = 250x$. The work required to stretch the spring from 20 cm to 25 cm $[25 - 20 = 5\,\text{cm} = 0.05\,\text{m}]$ is $W = \int_0^{0.05} 250x\,dx = \left[125x^2\right]_0^{0.05} = 125(0.0025) = 0.3125 \approx 0.31$ J.

7. (a) If $\int_0^{0.12} kx\,dx = 2$ J, then $2 = \left[\frac{1}{2}kx^2\right]_0^{0.12} = \frac{1}{2}k(0.0144) = 0.0072k$ and $k = \frac{2}{0.0072} = \frac{2500}{9} \approx 277.78$ N/m.

Thus, the work needed to stretch the spring from 35 cm to 40 cm is

$\int_{0.05}^{0.10} \frac{2500}{9}x\,dx = \left[\frac{1250}{9}x^2\right]_{1/20}^{1/10} = \frac{1250}{9}\left(\frac{1}{100} - \frac{1}{400}\right) = \frac{25}{24} \approx 1.04$ J.

(b) $f(x) = kx$, so $30 = \frac{2500}{9}x$ and $x = \frac{270}{2500}$ m $= 10.8$ cm

8. If $12 = \int_0^1 kx\,dx = \left[\frac{1}{2}kx^2\right]_0^1 = \frac{1}{2}k$, then $k = 24$ lb/ft and the work required is $\int_0^{3/4} 24x\,dx = \left[12x^2\right]_0^{3/4} = 12 \cdot \frac{9}{16} = \frac{27}{4} = 6.75$ ft-lb.

9. The distance from 20 cm to 30 cm is 0.1 m, so with $f(x) = kx$, we get $W_1 = \int_0^{0.1} kx\,dx = k\left[\frac{1}{2}x^2\right]_0^{0.1} = \frac{1}{200}k$.

Now $W_2 = \int_{0.1}^{0.2} kx\,dx = k\left[\frac{1}{2}x^2\right]_{0.1}^{0.2} = k\left(\frac{4}{200} - \frac{1}{200}\right) = \frac{3}{200}k$. Thus, $W_2 = 3W_1$.

10. Let L be the natural length of the spring in meters. Then

$6 = \int_{0.10-L}^{0.12-L} kx\,dx = \left[\frac{1}{2}kx^2\right]_{0.10-L}^{0.12-L} = \frac{1}{2}k\left[(0.12 - L)^2 - (0.10 - L)^2\right]$ and

$10 = \int_{0.12-L}^{0.14-L} kx\,dx = \left[\frac{1}{2}kx^2\right]_{0.12-L}^{0.14-L} = \frac{1}{2}k\left[(0.14 - L)^2 - (0.12 - L)^2\right]$.

Simplifying gives us $12 = k(0.0044 - 0.04L)$ and $20 = k(0.0052 - 0.04L)$. Subtracting the first equation from the second gives $8 = 0.0008k$, so $k = 10{,}000$. Now the second equation becomes $20 = 52 - 400L$, so $L = \frac{32}{400}$ m $= 8$ cm.

In Exercises 11 – 18, n is the number of subintervals of length Δx, and x_i^* is a sample point in the ith subinterval $[x_{i-1}, x_i]$.

11. (a) The portion of the rope from x ft to $(x + \Delta x)$ ft below the top of the building weighs $\frac{1}{2}\,\Delta x$ lb and must be lifted x_i^* ft, so its contribution to the total work is $\frac{1}{2}x_i^*\,\Delta x$ ft-lb. The total work is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{1}{2}x_i^*\,\Delta x = \int_0^{50} \tfrac{1}{2}x\,dx = \left[\tfrac{1}{4}x^2\right]_0^{50} = \tfrac{2500}{4} = 625 \text{ ft-lb}$$

Notice that the exact height of the building does not matter (as long as it is more than 50 ft).

(b) When half the rope is pulled to the top of the building, the work to lift the top half of the rope is $W_1 = \int_0^{25} \frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_0^{25} = \frac{625}{4}$ ft-lb. The bottom half of the rope is lifted 25 ft and the work needed to accomplish that is $W_2 = \int_{25}^{50} \frac{1}{2} \cdot 25\,dx = \frac{25}{2}\left[x\right]_{25}^{50} = \frac{625}{2}$ ft-lb. The total work done in pulling half the rope to the top of the building is $W = W_1 + W_2 = \frac{625}{2} + \frac{625}{4} = \frac{3}{4} \cdot 625 = \frac{1875}{4}$ ft-lb.

12. *Assumptions*:

1. After lifting, the chain is L-shaped, with 4 m of the chain lying along the ground.
2. The chain slides effortlessly and without friction along the ground while its end is lifted.
3. The weight density of the chain is constant throughout its length and therefore equals $(8\text{ kg/m})(9.8\text{ m/s}^2) = 78.4\text{ N/m}$.

The part of the chain x m from the lifted end is raised $6 - x$ m if $0 \le x \le 6$ m, and it is lifted 0 m if $x > 6$ m.

Thus, the work needed is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}(6 - x_i^*)\cdot 78.4\,\Delta x = \int_0^6 (6-x)78.4\,dx = 78.4\big[6x - \tfrac{1}{2}x^2\big]_0^6 = (78.4)(18) = 1411.2\text{ J}$$

13. The work needed to lift the cable is $\lim\limits_{n\to\infty}\sum_{i=1}^{n} 2x_i^*\,\Delta x = \int_0^{500} 2x\,dx = \big[x^2\big]_0^{500} = 250{,}000$ ft-lb. The work needed to lift the coal is 800 lb · 500 ft = 400,000 ft-lb. Thus, the total work required is 250,000 + 400,000 = 650,000 ft-lb.

14. The work needed to lift the bucket itself is 4 lb · 80 ft = 320 ft-lb. At time t (in seconds) the bucket is $x_i^* = 2t$ ft above its original 80 ft depth, but it now holds only $(40 - 0.2t)$ lb of water. In terms of distance, the bucket holds $\big[40 - 0.2\big(\tfrac{1}{2}x_i^*\big)\big]$ lb of water when it is x_i^* ft above its original 80 ft depth. Moving this amount of water a distance Δx requires $\big(40 - \tfrac{1}{10}x_i^*\big)\,\Delta x$ ft-lb of work. Thus, the work needed to lift the water is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}\big(40 - \tfrac{1}{10}x_i^*\big)\,\Delta x = \int_0^{80}\big(40 - \tfrac{1}{10}x\big)dx = \big[40x - \tfrac{1}{20}x^2\big]_0^{80} = (3200 - 320)\text{ ft-lb}$$

Adding the work of lifting the bucket gives a total of 3200 ft-lb of work.

15. At a height of x meters $(0 \le x \le 12)$, the mass of the rope is $(0.8\text{ kg/m})(12 - x\text{ m}) = (9.6 - 0.8x)$ kg and the mass of the water is $\big(\tfrac{36}{12}\text{ kg/m}\big)(12 - x\text{ m}) = (36 - 3x)$ kg. The mass of the bucket is 10 kg, so the total mass is $(9.6 - 0.8x) + (36 - 3x) + 10 = (55.6 - 3.8x)$ kg, and hence, the total force is $9.8(55.6 - 3.8x)$ N. The work needed to lift the bucket Δx m through the ith subinterval of $[0, 12]$ is $9.8(55.6 - 3.8x_i^*)\Delta x$, so the total work is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} 9.8(55.6 - 3.8x_i^*)\,\Delta x = \int_0^{12}(9.8)(55.6 - 3.8x)\,dx = 9.8\Big[55.6x - 1.9x^2\Big]_0^{12} = 9.8(393.6) \approx 3857\text{ J}$$

16. The chain's weight density is $\dfrac{25\text{ lb}}{10\text{ ft}} = 2.5$ lb/ft. The part of the chain x ft below the ceiling (for $5 \le x \le 10$) has to be lifted $2(x - 5)$ ft, so the work needed to lift the ith subinterval of the chain is $2(x_i^* - 5)(2.5\,\Delta x)$. The total work needed is

$$\begin{aligned} W &= \lim_{n\to\infty}\sum_{i=1}^{n} 2(x_i^* - 5)(2.5)\,\Delta x = \int_5^{10}[2(x-5)(2.5)]\,dx = 5\int_5^{10}(x-5)\,dx \\ &= 5\big[\tfrac{1}{2}x^2 - 5x\big]_5^{10} = 5\big[(50-50) - \big(\tfrac{25}{2} - 25\big)\big] = 5\big(\tfrac{25}{2}\big) = 62.5\text{ ft-lb} \end{aligned}$$

17. A "slice" of water Δx m thick and lying at a depth of x_i^* m (where $0 \le x_i^* \le \frac{1}{2}$) has volume $(2 \times 1 \times \Delta x)\text{ m}^3$, a mass of $2000\,\Delta x$ kg, weighs about $(9.8)(2000\,\Delta x) = 19{,}600\,\Delta x$ N, and thus requires about $19{,}600x_i^*\,\Delta x$ J of work for its removal.

So $W = \lim\limits_{n\to\infty}\sum_{i=1}^{n} 19{,}600x_i^*\,\Delta x = \int_0^{1/2} 19{,}600x\,dx = \big[9800x^2\big]_0^{1/2} = 2450$ J.

18. A horizontal cylindrical slice of water Δx ft thick has a volume of $\pi r^2 h = \pi \cdot 12^2 \cdot \Delta x$ ft^3 and weighs about $\left(62.5 \text{ lb/ft}^3\right)\left(144\pi\, \Delta x \text{ ft}^3\right) = 9000\pi\, \Delta x$ lb. If the slice lies x_i^* ft below the edge of the pool (where $1 \le x_i^* \le 5$), then the work needed to pump it out is about $9000\pi x_i^*\, \Delta x$. Thus,

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 9000\pi x_i^*\, \Delta x = \int_1^5 9000\pi x\, dx = \left[4500\pi x^2\right]_1^5 = 4500\pi(25-1) = 108{,}000\pi \text{ ft-lb}$$

19. A rectangular "slice" of water Δx m thick and lying x m above the bottom has width x m and volume $8x\,\Delta x$ m^3. It weighs about $(9.8 \times 1000)(8x\,\Delta x)$ N, and must be lifted $(5-x)$ m by the pump, so the work needed is about $(9.8 \times 10^3)(5-x)(8x\,\Delta x)$ J. The total work required is

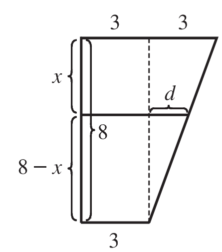
$$\begin{aligned} W &\approx \int_0^3 (9.8 \times 10^3)(5-x)8x\, dx = (9.8 \times 10^3) \int_0^3 (40x - 8x^2)\, dx = (9.8 \times 10^3)\left[20x^2 - \tfrac{8}{3}x^3\right]_0^3 \\ &= (9.8 \times 10^3)(180-72) = (9.8 \times 10^3)(108) = 1058.4 \times 10^3 \approx 1.06 \times 10^6 \text{ J} \end{aligned}$$

20. Let y measure depth (in meters) below the center of the spherical tank, so that $y = -3$ at the top of the tank and $y = -4$ at the spigot. A horizontal disk-shaped "slice" of water Δy m thick and lying at coordinate y has radius $\sqrt{9-y^2}$ m and volume $\pi r^2 \Delta y = \pi(9-y^2)\,\Delta y$ m^3. It weighs about $(9.8 \times 1000)\pi(9-y^2)\,\Delta y$ N and must be lifted $(y+4)$ m by the pump, so the work needed to pump it out is about $(9.8 \times 10^3)(y+4)\pi(9-y^2)\,\Delta y$ J. The total work required is

$$\begin{aligned} W &\approx \int_{-3}^3 (9.8 \times 10^3)(y+4)\pi(9-y^2)\, dy = (9.8 \times 10^3)\pi \int_{-3}^3 (9y - y^3 + 36 - 4y^2)\, dy \\ &= (9.8 \times 10^3)\pi(2)(4) \int_0^3 (9-y^2)\, dy \qquad \text{[by Theorem 5.5.6]} \\ &= (78.4 \times 10^3)\pi\left[9y - \tfrac{1}{3}y^3\right]_0^3 = (78.4 \times 10^3)\pi(18) = 1{,}411{,}200\pi \approx 4.43 \times 10^6 \text{ J} \end{aligned}$$

21. Let x measure depth (in feet) below the spout at the top of the tank. A horizontal disk-shaped "slice" of water Δx ft thick and lying at coordinate x has radius $\frac{3}{8}(16-x)$ ft $(\star)$ and volume $\pi r^2 \Delta x = \pi \cdot \frac{9}{64}(16-x)^2\,\Delta x$ ft^3. It weighs about $(62.5)\frac{9\pi}{64}(16-x)^2\,\Delta x$ lb and must be lifted x ft by the pump, so the work needed to pump it out is about $(62.5)x\,\frac{9\pi}{64}(16-x)^2\,\Delta x$ ft-lb. The total work required is

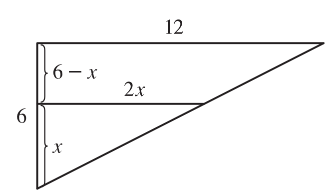
$$\begin{aligned} W &\approx \int_0^8 (62.5)x\,\tfrac{9\pi}{64}(16-x)^2\, dx = (62.5)\tfrac{9\pi}{64}\int_0^8 x(256 - 32x + x^2)\, dx \\ &= (62.5)\tfrac{9\pi}{64}\int_0^8 (256x - 32x^2 + x^3)\, dx = (62.5)\tfrac{9\pi}{64}\left[128x^2 - \tfrac{32}{3}x^3 + \tfrac{1}{4}x^4\right]_0^8 \\ &= (62.5)\frac{9\pi}{64}\left(\frac{11{,}264}{3}\right) = 33{,}000\pi \approx 1.04 \times 10^5 \text{ ft-lb} \end{aligned}$$



$(\star)$ From similar triangles, $\dfrac{d}{8-x} = \dfrac{3}{8}$.

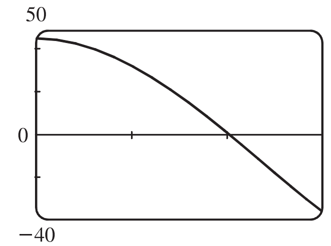
So $r = 3 + d = 3 + \frac{3}{8}(8-x) = \dfrac{3(8)}{8} + \dfrac{3}{8}(8-x) = \frac{3}{8}(16-x)$

22. Let x measure the distance (in feet) above the bottom of the tank. A horizontal "slice" of water Δx ft thick and lying at coordinate x has volume $10(2x)\,\Delta x$ ft^3. It weighs about $(62.5)20x\,\Delta x$ lb and must be lifted $(6-x)$ ft by the pump, so the work needed to pump it out is about $(62.5)(6-x)20x\,\Delta x$ ft-lb. The total work required is



$$W \approx \int_0^6 (62.5)(6-x)20x\, dx = 1250\int_0^6 (6x - x^2)\, dx = 1250\left[3x^2 - \tfrac{1}{3}x^3\right]_0^6 = 1250(36) = 45{,}000 \text{ ft-lb.}$$

23. If only 4.7×10^5 J of work is done, then only the water above a certain level (call it h) will be pumped out. So we use the same formula as in Exercise 19, except that the work is fixed, and we are trying to find the lower limit of integration:



$4.7 \times 10^5 \approx \int_h^3 (9.8 \times 10^3)(5 - x)8x\,dx = (9.8 \times 10^3)\left[20x^2 - \frac{8}{3}x^3\right]_h^3 \quad \Leftrightarrow$

$\frac{4.7}{9.8} \times 10^2 \approx 48 = \left(20 \cdot 3^2 - \frac{8}{3} \cdot 3^3\right) - \left(20h^2 - \frac{8}{3}h^3\right) \quad \Leftrightarrow$

$2h^3 - 15h^2 + 45 = 0$. To find the solution of this equation, we plot $2h^3 - 15h^2 + 45$ between $h = 0$ and $h = 3$. We see that the equation is satisfied for $h \approx 2.0$. So the depth of water remaining in the tank is about 2.0 m.

24. The only changes needed in the solution for Exercise 20 are: (1) change the lower limit from -3 to 0 and (2) change 1000 to 900.

$$\begin{aligned} W &\approx \int_0^3 (9.8 \times 900)(y+4)\pi(9 - y^2)\,dy = (9.8 \times 900)\,\pi \int_0^3 (9y - y^3 + 36 - 4y^2)\,dy \\ &= (9.8 \times 900)\pi\left[\tfrac{9}{2}y^2 - \tfrac{1}{4}y^4 + 36y - \tfrac{4}{3}y^3\right]_0^3 = (9.8 \times 900)\pi(92.25) = 813{,}645\pi \\ &\approx 2.56 \times 10^6 \text{ J} \quad [\text{about } 58\% \text{ of the work in Exercise 20}] \end{aligned}$$

25. $V = \pi r^2 x$, so V is a function of x and P can also be regarded as a function of x. If $V_1 = \pi r^2 x_1$ and $V_2 = \pi r^2 x_2$, then

$$\begin{aligned} W &= \int_{x_1}^{x_2} F(x)\,dx = \int_{x_1}^{x_2} \pi r^2 P(V(x))\,dx = \int_{x_1}^{x_2} P(V(x))\,dV(x) \qquad [\text{Let } V(x) = \pi r^2 x, \text{ so } dV(x) = \pi r^2\,dx.] \\ &= \int_{V_1}^{V_2} P(V)\,dV \quad \text{by the Substitution Rule.} \end{aligned}$$

26. $160 \text{ lb/in}^2 = 160 \cdot 144 \text{ lb/ft}^2$, $100 \text{ in}^3 = \frac{100}{1728} \text{ ft}^3$, and $800 \text{ in}^3 = \frac{800}{1728} \text{ ft}^3$.

$k = PV^{1.4} = (160 \cdot 144)\left(\frac{100}{1728}\right)^{1.4} = 23{,}040\left(\frac{25}{432}\right)^{1.4} \approx 426.5$. Therefore, $P \approx 426.5V^{-1.4}$ and

$$W = \int_{100/1728}^{800/1728} 426.5V^{-1.4}\,dV = 426.5\left[\frac{1}{-0.4}V^{-0.4}\right]_{25/432}^{25/54} = (426.5)(2.5)\left[\left(\tfrac{432}{25}\right)^{0.4} - \left(\tfrac{54}{25}\right)^{0.4}\right] \approx 1.88 \times 10^3 \text{ ft-lb.}$$

27. (a) $W = \displaystyle\int_a^b F(r)\,dr = \int_a^b G\frac{m_1 m_2}{r^2}\,dr = Gm_1m_2\left[\frac{-1}{r}\right]_a^b = Gm_1m_2\left(\frac{1}{a} - \frac{1}{b}\right)$

(b) By part (a), $W = GMm\left(\dfrac{1}{R} - \dfrac{1}{R + 1{,}000{,}000}\right)$ where M = mass of the earth in kg, R = radius of the earth in m, and m = mass of satellite in kg. (Note that 1000 km = 1,000,000 m.) Thus,

$$W = (6.67 \times 10^{-11})(5.98 \times 10^{24})(1000) \times \left(\frac{1}{6.37 \times 10^6} - \frac{1}{7.37 \times 10^6}\right) \approx 8.50 \times 10^9 \text{ J}$$

28. (a) $W = \displaystyle\int_R^\infty \frac{GMm}{r^2}\,dr = \lim_{t\to\infty}\int_R^t \frac{GMm}{r^2}\,dr = \lim_{t\to\infty} GMm\left[\frac{-1}{r}\right]_R^t = GMm\lim_{t\to\infty}\left(\frac{-1}{t} + \frac{1}{R}\right) = \frac{GMm}{R}$,

where M = mass of the earth = 5.98×10^{24} kg, m = mass of the satellite = 10^3 kg, R = radius of the earth = 6.37×10^6 m, and G = gravitational constant = 6.67×10^{-11} N·m²/kg².

Therefore, work $= \dfrac{6.67 \times 10^{-11} \cdot 5.98 \times 10^{24} \cdot 10^3}{6.37 \times 10^6} \approx 6.26 \times 10^{10}$ J.

(b) From part (a), $W = \dfrac{GMm}{R}$. The initial kinetic energy supplies the needed work,

so $\frac{1}{2}mv_0^2 = \dfrac{GMm}{R} \quad \Rightarrow \quad v_0 = \sqrt{\dfrac{2GM}{R}}$.

29. The weight density of water is $\delta = 62.5\ \text{lb/ft}^3$.

(a) $P = \delta d \approx (62.5\ \text{lb/ft}^3)(3\ \text{ft}) = 187.5\ \text{lb/ft}^2$

(b) $F = PA \approx (187.5\ \text{lb/ft}^2)(5\ \text{ft})(2\ \text{ft}) = 1875\ \text{lb}$. ($A$ is the area of the bottom of the tank.)

(c) As in Example 1, the area of the ith strip is $2\,(\Delta x)$ and the pressure is $\delta d = \delta x_i$. Thus,

$$F = \int_0^3 \delta x \cdot 2\,dx \approx (62.5)(2)\int_0^3 x\,dx = 125\left[\tfrac{1}{2}x^2\right]_0^3 = 125\left(\tfrac{9}{2}\right) = 562.5\ \text{lb}.$$

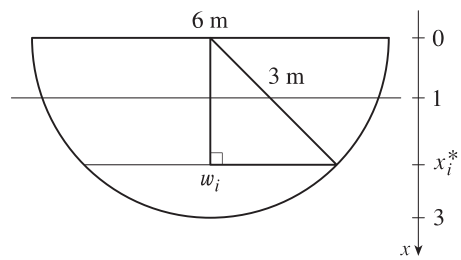
30. (a) $P = \rho g d = (820\ \text{kg/m}^3)(9.8\ \text{m/s}^2)(1.5\ \text{m}) = 12{,}054\ \text{Pa} \approx 12\ \text{kPa}$

(b) $F = PA = (12{,}054\ \text{Pa})(8\ \text{m})(4\ \text{m}) \approx 3.86 \times 10^5\ \text{N}$ (A is the area at the bottom of the tank.)

(c) The area of the ith strip is $4(\Delta x)$ and the pressure is $\rho g d = \rho g\, x_i$. Thus,

$$F = \int_0^{1.5} \rho g x \cdot 4\,dx = (820)(9.8)\cdot 4\int_0^{1.5} x\,dx = 32{,}144\left[\tfrac{1}{2}x^2\right]_0^{3/2} = 16{,}072\left(\tfrac{9}{4}\right) \approx 3.62 \times 10^4\ \text{N}.$$

31. Set up a vertical x-axis as shown. The base of the triangle shown in the figure has length $\sqrt{3^2 - (x_i^*)^2}$, so $w_i = 2\sqrt{9 - (x_i^*)^2}$, and the area of the ith rectangular strip is $2\sqrt{9 - (x_i^*)^2}\,\Delta x$. The ith rectangular strip is $(x_i^* - 1)$ m below the surface level of the water, so the pressure on the strip is $\rho g(x_i^* - 1)$. The hydrostatic force on the strip is $\rho g(x_i^* - 1)\cdot 2\sqrt{9 - (x_i^*)^2}\,\Delta x$ and the total force on the plate $\approx \sum_{i=1}^{n} \rho g(x_i^* - 1)\cdot 2\sqrt{9 - (x_i^*)^2}\,\Delta x$. The total force



$$F = \lim \sum_{i=1}^{n} \rho g(x_i^* - 1)\cdot 2\sqrt{9 - (x_i^*)^2}\,\Delta x = 2\rho g\int_1^3 (x-1)\sqrt{9 - x^2}\,dx$$

$$= 2\rho g\int_1^3 x\sqrt{9 - x^2}\,dx - 2\rho g\int_1^3 \sqrt{9 - x^2}\,dx \overset{30}{=} 2\rho g\left[-\tfrac{1}{3}(9 - x^2)^{3/2}\right]_1^3 - 2\rho g\left[\frac{x}{2}\sqrt{9 - x^2} + \tfrac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_1^3$$

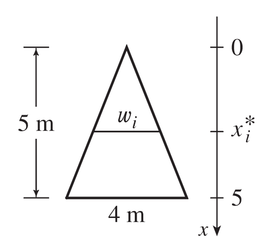
$$= 2\rho g\left[0 + \tfrac{1}{3}\left(8\sqrt{8}\right)\right] - 2\rho g\left[\left(0 + \tfrac{9}{2}\cdot\tfrac{\pi}{2}\right) - \left(\tfrac{1}{2}\sqrt{8} + \tfrac{9}{2}\sin^{-1}\left(\tfrac{1}{3}\right)\right)\right]$$

$$= \tfrac{32}{3}\sqrt{2}\,\rho g - \tfrac{9\pi}{2}\rho g + 2\sqrt{2}\,\rho g + 9\left[\sin^{-1}\left(\tfrac{1}{3}\right)\right]\rho g = \left(\tfrac{38}{3}\sqrt{2} - \tfrac{9\pi}{2} + 9\sin^{-1}\left(\tfrac{1}{3}\right)\right)\rho g$$

$$\approx 6.835 \cdot 1000 \cdot 9.8 \approx 6.7 \times 10^4\ \text{N}$$

Note: If you set up a typical coordinate system with the water level at $y = -1$, then $F = \int_{-3}^{-1} \rho g(-1 - y)2\sqrt{9 - y^2}\,dy$.

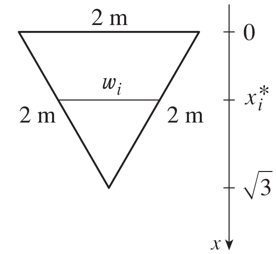
32. By similar triangles, $w_i/4 = x_i^*/5$, so $w_i = \frac{4}{5}x_i^*$ and the area of the ith strip is $\frac{4}{5}x_i^*\,\Delta x$. The pressure on the strip is $\rho g x_i^*$, so the hydrostatic force on the strip is $\rho g x_i^* \cdot \frac{4}{5}x_i^*\,\Delta x$ and the total force on the plate $\approx \sum_{i=1}^{n} \rho g x_i^* \cdot \frac{4}{5}x_i^*\,\Delta x$. The total force



$$F = \lim_{n\to\infty}\sum_{i=1}^{n} \rho g x_i^* \cdot \tfrac{4}{5}x_i^*\,\Delta x = \int_0^5 \rho g x \cdot \tfrac{4}{5}x\,dx = \tfrac{4}{5}\rho g\left[\tfrac{1}{3}x^3\right]_0^5 = \tfrac{4}{5}\rho g \cdot \tfrac{125}{3} = \tfrac{100}{3}\rho g$$

$$\approx \tfrac{100}{3}\cdot 1000 \cdot 9.8 \approx 3.3 \times 10^5\ \text{N}$$

33. Set up a vertical x-axis as shown. Then the area of the ith rectangular strip is $\left(2 - \frac{2}{\sqrt{3}}\,x_i^*\right)\Delta x$. $\left[\text{By similar triangles, } \frac{w_i}{2} = \frac{\sqrt{3} - x_i^*}{\sqrt{3}}, \text{ so } w_i = 2 - \frac{2}{\sqrt{3}}\,x_i^*.\right]$



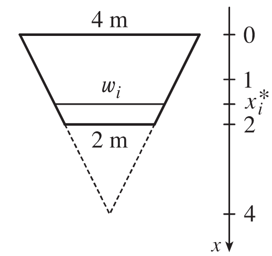
The pressure on the strip is $\rho g x_i^*$, so the hydrostatic force on the strip is $\rho g x_i^* \left(2 - \frac{2}{\sqrt{3}}\,x_i^*\right)\Delta x$ and the hydrostatic force on the plate $\approx \sum_{i=1}^{n} \rho g x_i^* \left(2 - \frac{2}{\sqrt{3}}\,x_i^*\right)\Delta x$.

The total force

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i^* \left(2 - \frac{2}{\sqrt{3}}\,x_i^*\right)\Delta x = \int_0^{\sqrt{3}} \rho g x \left(2 - \frac{2}{\sqrt{3}}\,x\right) dx = \rho g \int_0^{\sqrt{3}} \left(2x - \frac{2}{\sqrt{3}}\,x^2\right) dx$$

$$= \rho g \left[x^2 - \frac{2}{3\sqrt{3}}\,x^3\right]_0^{\sqrt{3}} = \rho g\,[(3-2)-0] = \rho g \approx 1000 \cdot 9.8 = 9.8 \times 10^3 \text{ N}$$

34. Set up a vertical x-axis as shown. Then the area of the ith rectangular strip is $(4 - x_i^*)\,\Delta x$. $\left[\text{By similar triangles, } \frac{w_i}{4} = \frac{4 - x_i^*}{4}, \text{ so } w_i = 4 - x_i^*.\right]$ The ith rectangular strip is $(x_i^* - 1)$ m below the surface level of the water, so the pressure on the strip is $\rho g(x_i^* - 1)$. The hydrostatic force on the strip is $\rho g(x_i^* - 1)(4 - x_i^*)\,\Delta x$ and the hydrostatic force on the plate $\approx \sum_{i=1}^{n} \rho g(x_i^* - 1)(4 - x_i^*)\,\Delta x$. The total force



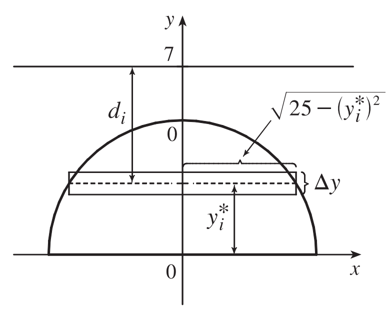
$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g(x_i^* - 1)(4 - x_i^*)\,\Delta x = \int_1^2 \rho g(x-1)(4-x)\,dx = \rho g \int_1^2 (-x^2 + 5x - 4)\,dx$$

$$= \rho g\left[-\tfrac{1}{3}x^3 + \tfrac{5}{2}x^2 - 4x\right]_1^2 = \rho g\left[\left(-\tfrac{8}{3} + 10 - 8\right) - \left(-\tfrac{1}{3} + \tfrac{5}{2} - 4\right)\right]$$

$$= \tfrac{7}{6}\rho g \approx \tfrac{7}{6} \cdot 1000 \cdot 9.8 \approx 1.14 \times 10^4 \text{ N}$$

Note: If you let the water level correspond to $x = 0$, then $F = \int_0^1 \rho g x(3 - x)\,dx$.

35. Set up coordinate axes as shown in the figure. The length of the ith strip is $2\sqrt{25 - (y_i^*)^2}$ and its area is $2\sqrt{25 - (y_i^*)^2}\,\Delta y$. The pressure on this strip is approximately $\delta d_i = 62.5(7 - y_i^*)$ and so the force on the strip is approximately $62.5(7 - y_i^*)2\sqrt{25 - (y_i^*)^2}\,\Delta y$. The total force

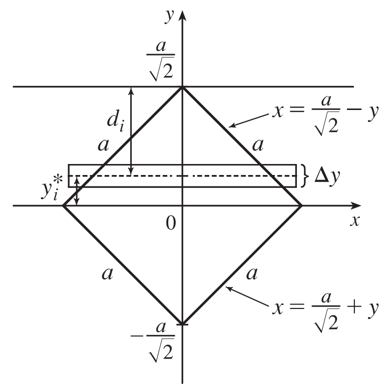


$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 62.5(7 - y_i^*)2\sqrt{25 - (y_i^*)^2}\,\Delta y = 125\int_0^5 (7 - y)\sqrt{25 - y^2}\,dy$$

$$= 125\left\{\int_0^5 7\sqrt{25 - y^2}\,dy - \int_0^5 y\sqrt{25 - y^2}\,dy\right\} = 125\left\{7\int_0^5 \sqrt{25 - y^2}\,dy - \left[-\tfrac{1}{3}(25 - y^2)^{3/2}\right]_0^5\right\}$$

$$= 125\left\{7\left(\tfrac{1}{4}\pi \cdot 5^2\right) + \tfrac{1}{3}(0 - 125)\right\} = 125\left(\tfrac{175\pi}{4} - \tfrac{125}{3}\right) \approx 11{,}972 \approx 1.2 \times 10^4 \text{ lb}$$

36. Set up coordinate axes as shown in the figure. For the *top half,* the length of the ith strip is $2(a/\sqrt{2} - y_i^*)$ and its area is $2(a/\sqrt{2} - y_i^*)\,\Delta y$. The pressure on this strip is approximately $\delta d_i = \delta(a/\sqrt{2} - y_i^*)$ and so the force on the strip is approximately $2\delta(a/\sqrt{2} - y_i^*)^2\,\Delta y$. The total force



$$F_1 = \lim_{n\to\infty}\sum_{i=1}^{n} 2\delta\left(\frac{a}{\sqrt{2}} - y_i^*\right)^2 \Delta y = 2\delta\int_0^{a/\sqrt{2}}\left(\frac{a}{\sqrt{2}} - y\right)^2 dy$$

$$= 2\delta\left[-\frac{1}{3}\left(\frac{a}{\sqrt{2}} - y\right)^3\right]_0^{a/\sqrt{2}} = -\frac{2}{3}\delta\left[0 - \left(\frac{a}{\sqrt{2}}\right)^3\right] = \frac{2\delta}{3}\frac{a^3}{2\sqrt{2}} = \frac{\sqrt{2}\,a^3\delta}{6}$$

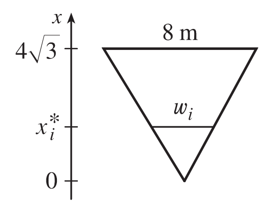
For the *bottom half,* the length is $2(a/\sqrt{2} + y_i^*)$ and the total force is

$$F_2 = \lim_{n\to\infty}\sum_{i=1}^{n} 2\delta\left(\frac{a}{\sqrt{2}} + y_i^*\right)\left(\frac{a}{\sqrt{2}} - y_i^*\right)\Delta y = 2\delta\int_{-a/\sqrt{2}}^{0}\left(\frac{a^2}{2} - y^2\right)dy = 2\delta\left[\tfrac{1}{2}a^2 y - \tfrac{1}{3}y^3\right]_{-a/\sqrt{2}}^{0}$$

$$= 2\delta\left[0 - \left(-\frac{\sqrt{2}\,a^3}{4} + \frac{\sqrt{2}\,a^3}{12}\right)\right] = 2\delta\left(\frac{\sqrt{2}\,a^3}{6}\right) = \frac{2\sqrt{2}\,a^3\delta}{6} \qquad [F_2 = 2F_1]$$

Thus, the total force $F = F_1 + F_2 = \dfrac{3\sqrt{2}\,a^3\delta}{6} = \dfrac{\sqrt{2}\,a^3\delta}{2}$.

37. By similar triangles, $\dfrac{8}{4\sqrt{3}} = \dfrac{w_i}{x_i^*} \;\Rightarrow\; w_i = \dfrac{2x_i^*}{\sqrt{3}}$. The area of the ith rectangular strip is $\dfrac{2x_i^*}{\sqrt{3}}\,\Delta x$ and the pressure on it is $\rho g(4\sqrt{3} - x_i^*)$.

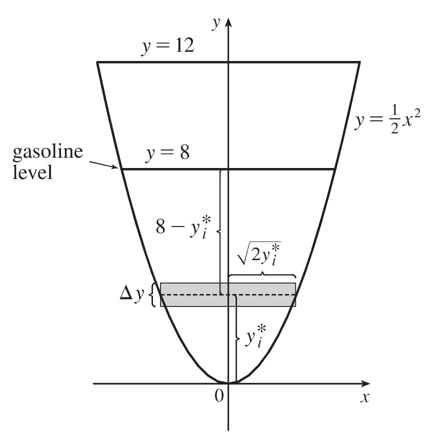


$$F = \int_0^{4\sqrt{3}} \rho g\left(4\sqrt{3} - x\right)\frac{2x}{\sqrt{3}}\,dx = 8\rho g\int_0^{4\sqrt{3}} x\,dx - \frac{2\rho g}{\sqrt{3}}\int_0^{4\sqrt{3}} x^2\,dx$$

$$= 4\rho g\left[x^2\right]_0^{4\sqrt{3}} - \frac{2\rho g}{3\sqrt{3}}\left[x^3\right]_0^{4\sqrt{3}} = 192\rho g - \frac{2\rho g}{3\sqrt{3}}\,64\cdot 3\sqrt{3} = 192\rho g - 128\rho g = 64\rho g$$

$$\approx 64(840)(9.8) \approx 5.27\times 10^5\ \text{N}$$

38. The area of the ith rectangular strip is $2\sqrt{2y_i^*}\,\Delta y$ and the pressure on it is $\delta d_i = \delta(8 - y_i^*)$.



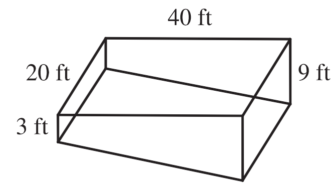
$$F = \int_0^8 \delta(8 - y)\,2\sqrt{2y}\,dy = 42\cdot 2\cdot\sqrt{2}\int_0^8 (8 - y)y^{1/2}\,dy$$

$$= 84\sqrt{2}\int_0^8 (8y^{1/2} - y^{3/2})\,dy = 84\sqrt{2}\left[8\cdot\tfrac{2}{3}y^{3/2} - \tfrac{2}{5}y^{5/2}\right]_0^8$$

$$= 84\sqrt{2}\left[8\cdot\tfrac{2}{3}\cdot 16\sqrt{2} - \tfrac{2}{5}\cdot 128\sqrt{2}\right]$$

$$= 84\sqrt{2}\cdot 256\sqrt{2}\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = 43{,}008\cdot\tfrac{2}{15} = 5734.4\ \text{lb}$$

39. (a) The area of a strip is $20\,\Delta x$ and the pressure on it is δx_i.



$$F = \int_0^3 \delta x 20\,dx = 20\delta\left[\tfrac{1}{2}x^2\right]_0^3 = 20\delta \cdot \tfrac{9}{2} = 90\delta$$
$$= 90(62.5) = 5625 \text{ lb} \approx 5.63 \times 10^3 \text{ lb}$$

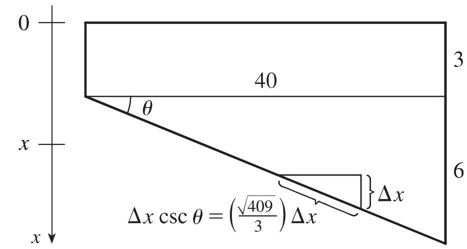
(b) $F = \int_0^9 \delta x 20\,dx = 20\delta\left[\tfrac{1}{2}x^2\right]_0^9 = 20\delta \cdot \tfrac{81}{2} = 810\delta = 810(62.5) = 50{,}625 \text{ lb} \approx 5.06 \times 10^4$ lb.

(c) For the first 3 ft, the length of the side is constant at 40 ft. For $3 < x \le 9$, we can use similar triangles to find the length a:

$$\frac{a}{40} = \frac{9-x}{6} \quad \Rightarrow \quad a = 40 \cdot \frac{9-x}{6}.$$

$$F = \int_0^3 \delta x 40\,dx + \int_3^9 \delta x(40)\,\frac{9-x}{6}\,dx = 40\delta\left[\tfrac{1}{2}x^2\right]_0^3 + \tfrac{20}{3}\delta\int_3^9 (9x - x^2)\,dx = 180\delta + \tfrac{20}{3}\delta\left[\tfrac{9}{2}x^2 - \tfrac{1}{3}x^3\right]_3^9$$
$$= 180\delta + \tfrac{20}{3}\delta\left[\left(\tfrac{729}{2} - 243\right) - \left(\tfrac{81}{2} - 9\right)\right] = 180\delta + 600\delta = 780\delta = 780(62.5) = 48{,}750 \text{ lb} \approx 4.88 \times 10^4 \text{ lb}$$

(d) For any right triangle with hypotenuse on the bottom,

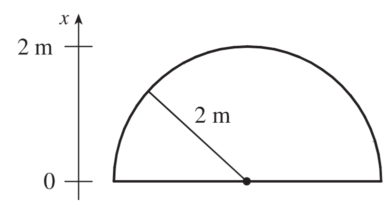


$$\sin\theta = \frac{\Delta x}{\text{hypotenuse}} \quad \Rightarrow$$

$$\text{hypotenuse} = \Delta x \csc\theta = \Delta x\,\frac{\sqrt{40^2 + 6^2}}{6} = \frac{\sqrt{409}}{3}\,\Delta x.$$

$$F = \int_3^9 \delta x 20\,\frac{\sqrt{409}}{3}\,dx = \tfrac{1}{3}\left(20\sqrt{409}\right)\delta\left[\tfrac{1}{2}x^2\right]_3^9$$
$$= \tfrac{1}{3} \cdot 10\sqrt{409}\,\delta(81 - 9) \approx 303{,}356 \text{ lb} \approx 3.03 \times 10^5 \text{ lb}$$

40. $F = \int_0^2 \rho g(10 - x)2\sqrt{4 - x^2}\,dx$



$$= 20\rho g\int_0^2 \sqrt{4 - x^2}\,dx - \rho g\int_0^2 \sqrt{4 - x^2}\,2x\,dx$$
$$= 20\rho g\tfrac{1}{4}\pi(2^2) - \rho g\int_0^4 u^{1/2}\,du \qquad [u = 4 - x^2,\, du = -2x\,dx]$$
$$= 20\pi\rho g - \tfrac{2}{3}\rho g\left[u^{3/2}\right]_0^4 = 20\pi\rho g - \tfrac{16}{3}\rho g = \rho g\left(20\pi - \tfrac{16}{3}\right)$$
$$= (1000)(9.8)\left(20\pi - \tfrac{16}{3}\right) \approx 5.63 \times 10^5 \text{ N}$$

41. $F = \int_2^5 \rho g x \cdot w(x)\,dx$, where $w(x)$ is the width of the plate at depth x. Since $n = 6$, $\Delta x = \frac{5-2}{6} = \frac{1}{2}$, and

$$F \approx S_6$$
$$= \rho g \cdot \tfrac{1/2}{3}[2 \cdot w(2) + 4 \cdot 2.5 \cdot w(2.5) + 2 \cdot 3 \cdot w(3) + 4 \cdot 3.5 \cdot w(3.5) + 2 \cdot 4 \cdot w(4) + 4 \cdot 4.5 \cdot w(4.5) + 5 \cdot w(5)]$$
$$= \tfrac{1}{6}\rho g(2 \cdot 0 + 10 \cdot 0.8 + 6 \cdot 1.7 + 14 \cdot 2.4 + 8 \cdot 2.9 + 18 \cdot 3.3 + 5 \cdot 3.6)$$
$$= \tfrac{1}{6}(1000)(9.8)(152.4) \approx 2.5 \times 10^5 \text{ N}$$

42. $M = m_1x_1 + m_2x_2 + m_3x_3 = 25(-2) + 20(3) + 10(7) = 80;\ \overline{x} = M/(m_1 + m_2 + m_3) = \frac{80}{55} = \frac{16}{11}$.

43. $m = \sum_{i=1}^{3} m_i = 6 + 5 + 10 = 21.$

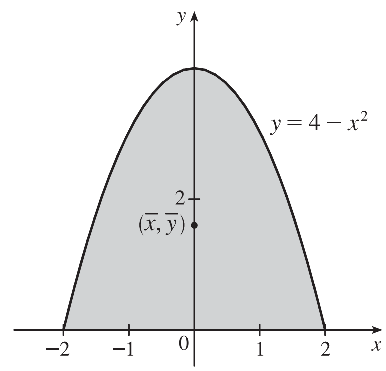
$M_x = \sum_{i=1}^{3} m_i\,y_i = 6(5) + 5(-2) + 10(-1) = 10;\ M_y = \sum_{i=1}^{3} m_i\,x_i = 6(1) + 5(3) + 10(-2) = 1.$

$\overline{x} = \dfrac{M_y}{m} = \dfrac{1}{21}$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{10}{21}$, so the center of mass of the system is $\left(\frac{1}{21}, \frac{10}{21}\right)$.

44. $M_x = \sum_{i=1}^{4} m_i\, y_i = 6(-2) + 5(4) + 1(-7) + 4(-1) = -3$, $M_y = \sum_{i=1}^{4} m_i\, x_i = 6(1) + 5(3) + 1(-3) + 4(6) = 42$,

and $m = \sum_{i=1}^{4} m_i = 16$, so $\overline{x} = \dfrac{M_y}{m} = \dfrac{42}{16} = \dfrac{21}{8}$ and $\overline{y} = \dfrac{M_x}{m} = -\dfrac{3}{16}$; the center of mass is $(\overline{x}, \overline{y}) = \left(\frac{21}{8}, -\frac{3}{16}\right)$.

45. Since the region in the figure is symmetric about the y-axis, we know that $\overline{x} = 0$. The region is "bottom-heavy," so we know that $\overline{y} < 2$, and we might guess that $\overline{y} = 1.5$.



$A = \int_{-2}^{2} (4 - x^2)\, dx = 2\int_0^2 (4 - x^2)\, dx = 2\left[4x - \frac{1}{3}x^3\right]_0^2$

$= 2\left(8 - \frac{8}{3}\right) = \frac{32}{3}$.

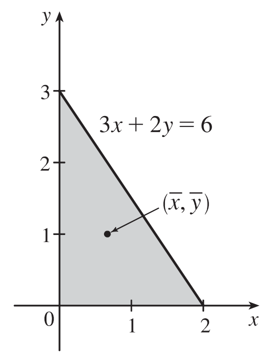
$\overline{x} = \frac{1}{A}\int_{-2}^{2} x(4 - x^2)\, dx = 0$ since $f(x) = x(4 - x^2)$ is an odd function (or since the region is symmetric about the y-axis).

$$\overline{y} = \frac{1}{A}\int_{-2}^{2} \frac{1}{2}(4 - x^2)^2\, dx = \frac{3}{32} \cdot \frac{1}{2} \cdot 2\int_0^2 (16 - 8x^2 + x^4)\, dx = \frac{3}{32}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2$$

$$= \frac{3}{32}\left(32 - \frac{64}{3} + \frac{32}{5}\right) = 3\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 3\left(\frac{8}{15}\right) = \frac{8}{5}$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{8}{5}\right)$.

46. The region in the figure is "left-heavy" and "bottom-heavy," so we know $\overline{x} < 1$ and $\overline{y} < 1.5$, and we might guess that $\overline{x} = 0.7$ and $\overline{y} = 1.2$.



$3x + 2y = 6 \quad \Leftrightarrow \quad 2y = 6 - 3x \quad \Leftrightarrow \quad y = 3 - \frac{3}{2}x$.

$A = \int_0^2 \left(3 - \frac{3}{2}x\right) dx = \left[3x - \frac{3}{4}x^2\right]_0^2 = 6 - 3 = 3$.

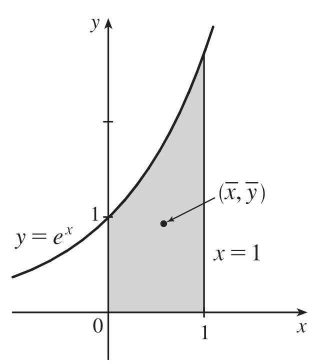
$\overline{x} = \frac{1}{A}\int_0^2 x\left(3 - \frac{3}{2}x\right) dx = \frac{1}{3}\int_0^2 \left(3x - \frac{3}{2}x^2\right) dx = \frac{1}{3}\left[\frac{3}{2}x^2 - \frac{1}{2}x^3\right]_0^2$

$= \frac{1}{3}(6 - 4) = \frac{2}{3}$.

$\overline{y} = \frac{1}{A}\int_0^2 \frac{1}{2}\left(3 - \frac{3}{2}x\right)^2 dx = \frac{1}{3} \cdot \frac{1}{2}\int_0^2 \left(9 - 9x + \frac{9}{4}x^2\right) dx = \frac{1}{6}\left[9x - \frac{9}{2}x^2 + \frac{3}{4}x^3\right]_0^2 = \frac{1}{6}(18 - 18 + 6) = 1$.

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{2}{3}, 1\right)$.

47. The region in the figure is "right-heavy" and "bottom-heavy," so we know $\overline{x} > 0.5$ and $\overline{y} < 1$, and we might guess that $\overline{x} = 0.6$ and $\overline{y} = 0.9$.



$A = \int_0^1 e^x\, dx = \left[e^x\right]_0^1 = e - 1$.

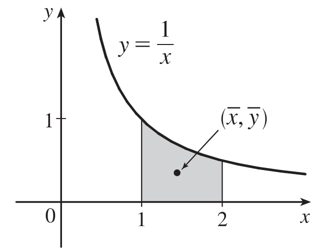
$\overline{x} = \frac{1}{A}\int_0^1 xe^x\, dx = \frac{1}{e - 1}\left[xe^x - e^x\right]_0^1$ [by parts]

$= \frac{1}{e - 1}[0 - (-1)] = \frac{1}{e - 1}$.

$\overline{y} = \frac{1}{A}\int_0^1 \frac{1}{2}(e^x)^2\, dx = \frac{1}{e - 1} \cdot \frac{1}{4}\left[e^{2x}\right]_0^1 = \frac{1}{4(e - 1)}\left(e^2 - 1\right) = \frac{e + 1}{4}$.

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{1}{e - 1}, \frac{e + 1}{4}\right) \approx (0.58, 0.93)$.

48. The region in the figure is "left-heavy" and "bottom-heavy," so we know $\overline{x} < 1.5$ and $\overline{y} < 0.5$, and we might guess that $\overline{x} = 1.4$ and $\overline{y} = 0.4$.



$A = \int_1^2 \frac{1}{x}\,dx = [\ln x]_1^2 = \ln 2$. $\overline{x} = \frac{1}{A}\int_1^2 x \cdot \frac{1}{x}\,dx = \frac{1}{A}[x]_1^2 = \frac{1}{A} = \frac{1}{\ln 2}$.

$\overline{y} = \frac{1}{A}\int_1^2 \frac{1}{2}\left(\frac{1}{x}\right)^2\,dx = \frac{1}{2A}\int_1^2 x^{-2}\,dx = \frac{1}{2A}\left[-\frac{1}{x}\right]_1^2$

$= \frac{1}{2\ln 2}\left(-\frac{1}{2} + 1\right) = \frac{1}{4\ln 2}$.

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{1}{\ln 2}, \frac{1}{4\ln 2}\right) \approx (1.44, 0.36)$.

49. The line has equation $y = \frac{3}{4}x$. $A = \frac{1}{2}(4)(3) = 6$, so $m = \rho A = 10(6) = 60$.

$$M_x = \rho\int_0^4 \frac{1}{2}\left(\frac{3}{4}x\right)^2 dx = 10\int_0^4 \frac{9}{32}x^2\,dx = \frac{45}{16}\left[\frac{1}{3}x^3\right]_0^4 = \frac{45}{16}\left(\frac{64}{3}\right) = 60$$

$$M_y = \rho\int_0^4 x\left(\frac{3}{4}x\right)dx = \frac{15}{2}\int_0^4 x^2\,dx = \frac{15}{2}\left[\frac{1}{3}x^3\right]_0^4 = \frac{15}{2}\left(\frac{64}{3}\right) = 160$$

$\overline{x} = \dfrac{M_y}{m} = \dfrac{160}{60} = \dfrac{8}{3}$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{60}{60} = 1$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{8}{3}, 1\right)$.

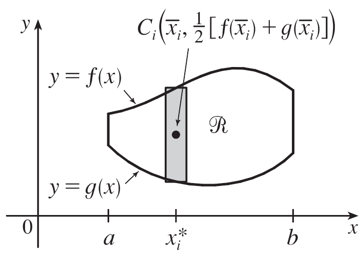
50. By symmetry about the line $y = x$, we expect that $\overline{x} = \overline{y}$. $A = \frac{1}{4}\pi r^2$, so $m = \rho A = 2A = \frac{1}{2}\pi r^2$.

$M_x = \rho\int_0^r \frac{1}{2}\left(\sqrt{r^2 - x^2}\right)^2 dx = 2 \cdot \frac{1}{2}\int_0^r (r^2 - x^2)\,dx = \left[r^2 x - \frac{1}{3}x^3\right]_0^r = \frac{2}{3}r^3$.

$M_y = \rho\int_0^r x\sqrt{r^2 - x^2}\,dx = \int_0^r (r^2 - x^2)^{1/2} 2x\,dx = \int_0^{r^2} u^{1/2}\,du \quad [u = r^2 - x^2] \;= \left[\frac{2}{3}u^{3/2}\right]_0^{r^2} = \frac{2}{3}r^3$.

$\overline{x} = \dfrac{1}{m}M_y = \dfrac{2}{\pi r^2}\left(\frac{2}{3}r^3\right) = \frac{4}{3\pi}r$, $\overline{y} = \dfrac{1}{m}M_x = \dfrac{2}{\pi r^2}\left(\frac{2}{3}r^3\right) = \frac{4}{3\pi}r$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\dfrac{4}{3\pi}r, \dfrac{4}{3\pi}r\right)$.

51. (a)



Suppose the region lies between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$, as illustrated in the figure. Use n subintervals determined by points x_i with $a = x_0 < x_1 < \cdots < x_n = b$ and choose $x_i^* = \overline{x}_i$ to be the midpoint of the ith subinterval; that is, $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Then the centroid of the ith approximating rectangle R_i is its center $C_i = \left(\overline{x}_i, \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)]\right)$.

Its area is $[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$, so its mass is $\rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$.

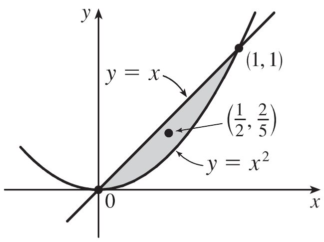
Thus, $M_y(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x \cdot \overline{x}_i = \rho\overline{x}_i[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$ and

$M_x(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x \cdot \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)] = \rho \cdot \frac{1}{2}\left\{[f(\overline{x}_i)]^2 - [g(\overline{x}_i)]^2\right\}\Delta x$. Summing over i and taking the limit as $n \to \infty$, we get $M_y = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} \rho\overline{x}_i[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x = \rho\int_a^b x[f(x) - g(x)]\,dx$ and

$M_x = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} \rho \cdot \frac{1}{2}\left[f(\overline{x}_i)^2 - g(\overline{x}_i)^2\right]\Delta x = \rho\int_a^b \frac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\}dx$. Thus,

$$\overline{x} = \frac{M_y}{m} = \frac{M_y}{\rho A} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx \text{ and } \overline{y} = \frac{M_x}{m} = \frac{M_x}{\rho A} = \frac{1}{A}\int_a^b \tfrac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\}dx.$$

(b)



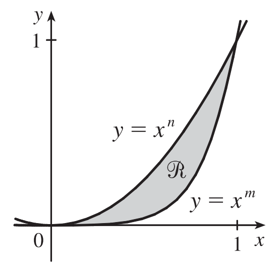
The region is sketched in the figure. We take $f(x) = x$, $g(x) = x^2$, $a = 0$, and $b = 1$ in the formulas in part (a). First we note that the area of the region is $A = \int_0^1 (x - x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{6}$.

Therefore, $\overline{x} = \frac{1}{A}\int_0^1 x[f(x) - g(x)]\,dx = \frac{1}{1/6}\int_0^1 x(x - x^2)\,dx = 6\int_0^1 (x^2 - x^3)\,dx = 6\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{2}$

and $\overline{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\}dx = \frac{1}{1/6}\int_0^1 \frac{1}{2}(x^2 - x^4)dx = 3\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \frac{2}{5}$.

The centroid is $\left(\frac{1}{2}, \frac{2}{5}\right)$.

52. (a) Let $0 \le x \le 1$. If $n < m$, then $x^n > x^m$; that is, raising x to a larger power produces a smaller number.



(b) Using Formulas 9 and the fact that the area of $\mathcal{R}$ is

$A = \int_0^1 (x^n - x^m)\,dx = \dfrac{1}{n+1} - \dfrac{1}{m+1} = \dfrac{m-n}{(n+1)(m+1)}$, we get

$$\overline{x} = \frac{(n+1)(m+1)}{m-n}\int_0^1 x[x^n - x^m]\,dx = \frac{(n+1)(m+1)}{m-n}\int_0^1 \left(x^{n+1} - x^{m+1}\right)dx$$
$$= \frac{(n+1)(m+1)}{m-n}\left[\frac{1}{n+2} - \frac{1}{m+2}\right] = \frac{(n+1)(m+1)}{(n+2)(m+2)}$$

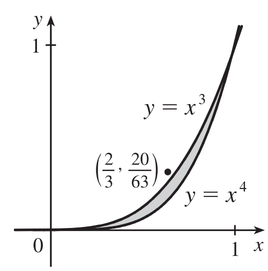
and

$$\overline{y} = \frac{(n+1)(m+1)}{m-n}\int_0^1 \tfrac{1}{2}\left[(x^n)^2 - (x^m)^2\right]dx = \frac{(n+1)(m+1)}{2(m-n)}\int_0^1 \left(x^{2n} - x^{2m}\right)dx$$
$$= \frac{(n+1)(m+1)}{2(m-n)}\left[\frac{1}{2n+1} - \frac{1}{2m+1}\right] = \frac{(n+1)(m+1)}{(2n+1)(2m+1)}$$

(c) If we take $n = 3$ and $m = 4$, then

$$(\overline{x}, \overline{y}) = \left(\frac{4\cdot 5}{5\cdot 6}, \frac{4\cdot 5}{7\cdot 9}\right) = \left(\frac{2}{3}, \frac{20}{63}\right)$$

which lies outside $\mathcal{R}$ since $\left(\frac{2}{3}\right)^3 = \frac{8}{27} < \frac{20}{63}$. This is the simplest of many possibilities.



DISCOVERY PROJECT Complementary Coffee Cups

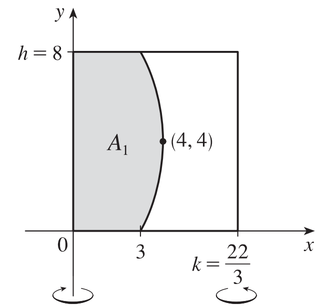
1. Cup A has volume $V_A = \int_0^h \pi[f(y)]^2\,dy$ and cup B has volume

$$V_B = \int_0^h \pi[k - f(y)]^2\,dy = \int_0^h \pi\{k^2 - 2kf(y) + [f(y)]^2\}\,dy$$
$$= \left[\pi k^2 y\right]_0^h - 2\pi k\int_0^h f(y)\,dy + \int_0^h \pi[f(y)]^2\,dy = \pi k^2 h - 2\pi k A_1 + V_A$$

Thus, $V_A = V_B \iff \pi k(kh - 2A_1) = 0 \iff k = 2(A_1/h)$; that is, k is twice the average value of f on the interval $[0, h]$.

2. From Problem 1, $V_A = V_B \iff kh = 2A_1 \iff A_1 + A_2 = 2A_1 \iff A_2 = A_1$.

3. We'll use a cup that is $h = 8$ cm high with a diameter of 6 cm on the top and the bottom and symmetrically bulging to a diameter of 8 cm in the middle (all inside dimensions).



For an equation, we'll use a parabola with a vertex at $(4, 4)$; that is, $x = a(y-4)^2 + 4$. To find a, use the point $(3, 0)$:
$3 = a(0-4)^2 + 4 \quad \Rightarrow \quad -1 = 16a \quad \Rightarrow \quad a = -\frac{1}{16}$. To find k, we'll use the relationship in Problem 1, so we need A_1.

$$A_1 = \int_0^8 \left[-\tfrac{1}{16}(y-4)^2 + 4\right] dy = \int_{-4}^4 \left(-\tfrac{1}{16}u^2 + 4\right) du \quad [u = y - 4]$$
$$= 2\int_0^4 \left(-\tfrac{1}{16}u^2 + 4\right) du = 2\left[-\tfrac{1}{48}u^3 + 4u\right]_0^4 = 2\left(-\tfrac{4}{3} + 16\right) = \tfrac{88}{3}.$$

Thus, $k = 2(A_1/h) = 2\left(\frac{88/3}{8}\right) = \frac{22}{3}$.

So with $h = 8$ and curve $x = -\frac{1}{16}(y-4)^2 + 4$, we have

$$V_A = \int_0^8 \pi\left[-\tfrac{1}{16}(y-4)^2 + 4\right]^2 dy = \pi\int_{-4}^4 \left(-\tfrac{1}{16}u^2 + 4\right)^2 du \quad [u = y - 4] \quad = 2\pi\int_0^4 \left(\tfrac{1}{256}u^4 - \tfrac{1}{2}u^2 + 16\right) du$$
$$= 2\pi\left[\tfrac{1}{1280}u^5 - \tfrac{1}{6}u^3 + 16u\right]_0^4 = 2\pi\left(\tfrac{4}{5} - \tfrac{32}{3} + 64\right) = 2\pi\left(\tfrac{812}{15}\right) = \tfrac{1624}{15}\pi$$

This is approximately 340 cm^3 or 11.5 fl. oz. And with $k = \frac{22}{3}$, we know from Problem 1 that cup B holds the same amount.

6.7 Applications to Economics and Biology

1. By the Net Change Theorem, $C(2000) - C(0) = \int_0^{2000} C'(x)\,dx \quad \Rightarrow$

$$C(2000) = 20{,}000 + \int_0^{2000}(5 - 0.008x + 0.000009x^2)\,dx = 20{,}000 + \left[5x - 0.004x^2 + 0.000003x^3\right]_0^{2000}$$
$$= 20{,}000 + 10{,}000 - 0.004(4{,}000{,}000) + 0.000003(8{,}000{,}000{,}000) = 30{,}000 - 16{,}000 + 24{,}000$$
$$= \$38{,}000$$

2. By the Net Change Theorem, $R(5000) - R(1000) = \int_{1000}^{5000} R'(x)\,dx \quad \Rightarrow$

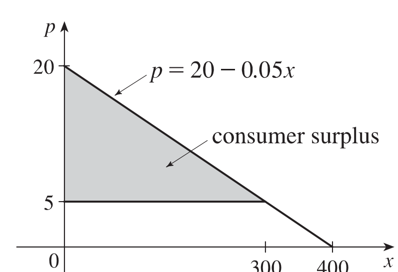
$$R(5000) = 12{,}400 + \int_{1000}^{5000}(12 - 0.0004x)\,dx = 12{,}400 + \left[12x - 0.0002x^2\right]_{1000}^{5000}$$
$$= 12{,}400 + (60{,}000 - 5{,}000) - (12{,}000 - 200) = \$55{,}600$$

3. If the production level is raised from 1200 units to 1600 units, then the increase in cost is

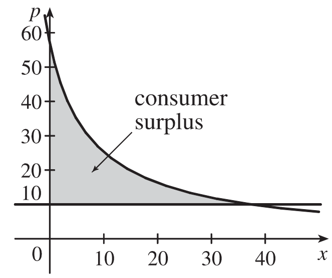
$$C(1600) - C(1200) = \int_{1200}^{1600} C'(x)\,dx = \int_{1200}^{1600}(74 + 1.1x - 0.002x^2 + 0.00004x^3)\,dx$$
$$= \left[74x + 0.55x^2 - \tfrac{0.002}{3}x^3 + 0.00001x^4\right]_{1200}^{1600} = 64{,}331{,}733.33 - 20{,}464{,}800 = \$43{,}866{,}933.33$$

4. Consumer surplus $= \int_0^{300} [p(x) - p(300)]\,dx$

$$= \int_0^{300}[20 - 0.05x - (5)]\,dx$$
$$= \int_0^{300}(15 - 0.05x)\,dx = \left[15x - 0.025x^2\right]_0^{300}$$
$$= 4500 - 2250 = \$2250$$

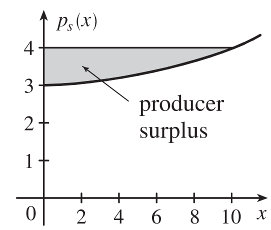


5. $p(x) = 10 \quad \Rightarrow \quad \dfrac{450}{x+8} = 10 \quad \Rightarrow \quad x + 8 = 45 \quad \Rightarrow \quad x = 37.$



$$\text{Consumer surplus} = \int_0^{37} [p(x) - 10]\,dx = \int_0^{37} \left(\frac{450}{x+8} - 10\right) dx$$
$$= \left[450 \ln(x+8) - 10x\right]_0^{37} = (450 \ln 45 - 370) - 450 \ln 8$$
$$= 450 \ln\left(\tfrac{45}{8}\right) - 370 \approx \$407.25$$

6. $p_S(x) = 3 + 0.01x^2. \quad P = p_S(10) = 3 + 1 = 4.$

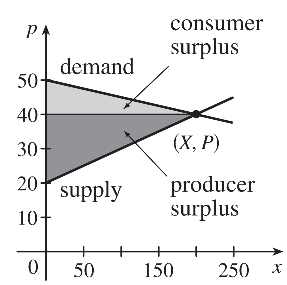


$$\text{Producer surplus} = \textstyle\int_0^{10} [P - p_S(x)]\,dx = \int_0^{10} \left[4 - 3 - 0.01x^2\right] dx$$
$$= \left[x - \tfrac{0.01}{3}x^3\right]_0^{10} \approx 10 - 3.33 = \$6.67$$

7. $P = p_S(x) \quad \Rightarrow \quad 400 = 200 + 0.2x^{3/2} \quad \Rightarrow \quad 200 = 0.2x^{3/2} \quad \Rightarrow \quad 1000 = x^{3/2} \quad \Rightarrow \quad x = 1000^{2/3} = 100.$

$$\text{Producer surplus} = \textstyle\int_0^{100} [P - p_S(x)]\,dx = \int_0^{100} [400 - (200 + 0.2x^{3/2})]\,dx = \int_0^{100} \left(200 - \frac{1}{5}x^{3/2}\right) dx$$
$$= \left[200x - \tfrac{2}{25}x^{5/2}\right]_0^{100} = 20{,}000 - 8{,}000 = \$12{,}000$$

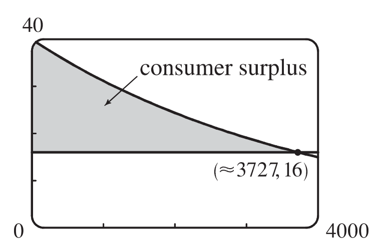
8. $p = 50 - \frac{1}{20}x$ and $p = 20 + \frac{1}{10}x$ intersect at $p = 40$ and $x = 200$.



$$\text{Consumer surplus} = \textstyle\int_0^{200} \left(50 - \frac{1}{20}x - 40\right) dx = \left[10x - \frac{1}{40}x^2\right]_0^{200} = \$1000$$

$$\text{Producer surplus} = \textstyle\int_0^{200} \left(40 - 20 - \frac{1}{10}x\right) dx = \left[20x - \frac{1}{20}x^2\right]_0^{200} = \$2000$$

9. $p(x) = \dfrac{800{,}000e^{-x/5000}}{x + 20{,}000} = 16 \quad \Rightarrow \quad x = x_1 \approx 3727.04.$



$$\text{Consumer surplus} = \textstyle\int_0^{x_1} [p(x) - 16]\,dx \approx \$37{,}753$$

10. The demand function is linear with slope $\frac{-0.5}{35} = -\frac{1}{70}$ and $p(400) = 7.5$, so an equation is $p - 7.5 = -\frac{1}{70}(x - 400)$ or $p = -\frac{1}{70}x + \frac{185}{14}$. A selling price of \$6 implies that $6 = -\frac{1}{70}x + \frac{185}{14} \quad \Rightarrow \quad \frac{1}{70}x = \frac{185}{14} - \frac{84}{14} = \frac{101}{14} \quad \Rightarrow \quad x = 505.$

$$\text{Consumer surplus} = \textstyle\int_0^{505} \left(-\frac{1}{70}x + \frac{185}{14} - 6\right) dx = \left[-\frac{1}{140}x^2 + \frac{101}{14}x\right]_0^{505} \approx \$1821.61.$$

11. $f(8) - f(4) = \int_4^8 f'(t)\,dt = \int_4^8 \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_4^8 = \frac{2}{3}\left(16\sqrt{2} - 8\right) \approx \$9.75$ million

12. The total revenue R obtained in the first four years is

$$R = \textstyle\int_0^4 f(t)\,dt = \int_0^4 9000\sqrt{1 + 2t}\,dt = \int_1^9 9000u^{1/2}\left(\frac{1}{2}\,du\right) \qquad [u = 1 + 2t,\, du = 2\,dt]$$
$$= 4500\left[\tfrac{2}{3}u^{3/2}\right]_1^9 = 3000(27 - 1) = \$78{,}000$$

13. $N = \int_a^b Ax^{-k}\,dx = A\left[\dfrac{x^{-k+1}}{-k+1}\right]_a^b = \dfrac{A}{1-k}\left(b^{1-k} - a^{1-k}\right)$.

Similarly, $\int_a^b Ax^{1-k}\,dx = A\left[\dfrac{x^{2-k}}{2-k}\right]_a^b = \dfrac{A}{2-k}\left(b^{2-k} - a^{2-k}\right)$.

Thus, $\overline{x} = \dfrac{1}{N}\int_a^b Ax^{1-k}\,dx = \dfrac{[A/(2-k)](b^{2-k} - a^{2-k})}{[A/(1-k)](b^{1-k} - a^{1-k})} = \dfrac{(1-k)(b^{2-k} - a^{2-k})}{(2-k)(b^{1-k} - a^{1-k})}$.

14. $n(9) - n(5) = \int_5^9 (2200 + 10e^{0.8t})\,dt = \left[2200t + \dfrac{10e^{0.8t}}{0.8}\right]_5^9 = \left[2200t\right]_5^9 + \frac{25}{2}\left[e^{0.8t}\right]_5^9$

$= 2200(9-5) + 12.5(e^{7.2} - e^4) \approx 24{,}860$

15. $F = \dfrac{\pi P R^4}{8\eta l} = \dfrac{\pi(4000)(0.008)^4}{8(0.027)(2)} \approx 1.19 \times 10^{-4}\ \text{cm}^3/\text{s}$

16. If the flux remains constant, then $\dfrac{\pi P_0 R_0^4}{8\eta l} = \dfrac{\pi P R^4}{8\eta l} \Rightarrow P_0 R_0^4 = PR^4 \Rightarrow \dfrac{P}{P_0} = \left(\dfrac{R_0}{R}\right)^4$.

$R = \frac{3}{4}R_0 \Rightarrow \dfrac{P}{P_0} = \left(\dfrac{R_0}{\frac{3}{4}R_0}\right)^4 \Rightarrow P = P_0\left(\frac{4}{3}\right)^4 \approx 3.1605P_0 > 3P_0$; that is, the blood pressure is more than tripled.

17. From (3), $F = \dfrac{A}{\int_0^T c(t)\,dt} = \dfrac{6}{20I}$, where

$$I = \int_0^{10} te^{-0.6t}\,dt = \left[\frac{1}{(-0.6)^2}\left(-0.6t - 1\right)e^{-0.6t}\right]_0^{10} \quad \begin{bmatrix}\text{integrating} \\ \text{by parts}\end{bmatrix} = \tfrac{1}{0.36}(-7e^{-6} + 1)$$

Thus, $F = \dfrac{6(0.36)}{20(1 - 7e^{-6})} = \dfrac{0.108}{1 - 7e^{-6}} \approx 0.1099\ \text{L/s or } 6.594\ \text{L/min}$.

18. As in Example 2, we will estimate the cardiac output using Simpson's Rule with $\Delta t = (20-0)/10 = 2$.

$$\begin{aligned}\int_0^{20} c(t)\,dt &\approx \tfrac{2}{3}[c(0) + 4c(2) + 2c(4) + 4c(6) + 2c(8) + 4c(10) + 2c(12) + 4c(14) + 2c(16) + 4c(18) + c(20)] \\ &= \tfrac{2}{3}[0 + 4(2.4) + 2(5.1) + 4(7.8) + 2(7.6) + 4(5.4) + 2(3.9) + 4(2.3) + 2(1.6) + 4(0.7) + 0] \\ &= \tfrac{2}{3}(110.8) \approx 73.87\ \text{mg}\cdot\text{s/L}\end{aligned}$$

Therefore, $F \approx \dfrac{A}{73.87} = \dfrac{8}{73.87} \approx 0.1083\ \text{L/s or } 6.498\ \text{L/min}$.

19. As in Example 2, we will estimate the cardiac output using Simpson's Rule with $\Delta t = (16-0)/8 = 2$.

$$\begin{aligned}\int_0^{16} c(t)\,dt &\approx \tfrac{2}{3}[c(0) + 4c(2) + 2c(4) + 4c(6) + 2c(8) + 4c(10) + 2c(12) + 4c(14) + c(16)] \\ &\approx \tfrac{2}{3}[0 + 4(6.1) + 2(7.4) + 4(6.7) + 2(5.4) + 4(4.1) + 2(3.0) + 4(2.1) + 1.5] \\ &= \tfrac{2}{3}(109.1) = 72.7\overline{3}\ \text{mg}\cdot\text{s/L}\end{aligned}$$

Therefore, $F \approx \dfrac{A}{72.7\overline{3}} = \dfrac{7}{72.7\overline{3}} \approx 0.0962\ \text{L/s or } 5.77\ \text{L/min}$.

6.8 Probability

1. (a) $\int_{30{,}000}^{40{,}000} f(x)\,dx$ is the probability that a randomly chosen tire will have a lifetime between 30,000 and 40,000 miles.

(b) $\int_{25{,}000}^{\infty} f(x)\,dx$ is the probability that a randomly chosen tire will have a lifetime of at least 25,000 miles.

2. (a) The probability that you drive to school in less than 15 minutes is $\int_0^{15} f(t)\,dt$.

(b) The probability that it takes you more than half an hour to get to school is $\int_{30}^{\infty} f(t)\,dt$.

3. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, (1) $f(x) \geq 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. For $0 \leq x \leq 4$, we have $f(x) = \frac{3}{64}x\sqrt{16 - x^2} \geq 0$, so $f(x) \geq 0$ for all x. Also,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^4 \tfrac{3}{64}x\sqrt{16-x^2}\,dx = -\tfrac{3}{128}\int_0^4 (16-x^2)^{1/2}(-2x)\,dx = -\tfrac{3}{128}\left[\tfrac{2}{3}(16-x^2)^{3/2}\right]_0^4$$
$$= -\tfrac{1}{64}\left[(16-x^2)^{3/2}\right]_0^4 = -\tfrac{1}{64}(0-64) = 1.$$

Therefore, f is a probability density function.

(b) $P(X < 2) = \int_{-\infty}^2 f(x)\,dx = \int_0^2 \frac{3}{64}x\sqrt{16-x^2}\,dx = -\frac{3}{128}\int_0^2 (16-x^2)^{1/2}(-2x)\,dx$
$$= -\tfrac{3}{128}\left[\tfrac{2}{3}(16-x^2)^{3/2}\right]_0^2 = -\tfrac{1}{64}\left[(16-x^2)^{3/2}\right]_0^2 = -\tfrac{1}{64}(12^{3/2} - 16^{3/2})$$
$$= \tfrac{1}{64}\left(64 - 12\sqrt{12}\right) = \tfrac{1}{64}\left(64 - 24\sqrt{3}\right) = 1 - \tfrac{3}{8}\sqrt{3} \approx 0.350481$$

4. (a) Since $f(x) = xe^{-x} \geq 0$ if $x \geq 0$ and $f(x) = 0$ if $x < 0$, it follows that $f(x) \geq 0$ for all x. Also,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} xe^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx \overset{96}{=} \text{[or by parts]} \lim_{t\to\infty}\left[(-x-1)e^{-x}\right]_0^t$$
$$= \lim_{t\to\infty}\left[(-t-1)e^{-t}+1\right] = 1 - \lim_{t\to\infty}\frac{t+1}{e^t} \overset{\text{H}}{=} 1 - \lim_{t\to\infty}\frac{1}{e^t} = 1 - 0 = 1$$

Thus, f is a probability density function.

(b) $P(1 \leq X \leq 2) = \int_1^2 xe^{-x}\,dx = \left[(-x-1)e^{-x}\right]_1^2 = -3e^{-2} + 2e^{-1} = 2/e - 3/e^2$ [≈ 0.33]

5. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, (1) $f(x) \geq 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. If $c \geq 0$, then $f(x) \geq 0$, so condition (1) is satisfied. For condition (2), we see that $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx$ and

$$\int_0^{\infty} \frac{c}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{c}{1+x^2}\,dx = c\lim_{t\to\infty}\left[\tan^{-1}x\right]_0^t = c\lim_{t\to\infty}\tan^{-1}t = c\left(\frac{\pi}{2}\right)$$

Similarly, $\displaystyle\int_{-\infty}^0 \frac{c}{1+x^2}\,dx = c\left(\frac{\pi}{2}\right)$, so $\displaystyle\int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = 2c\left(\frac{\pi}{2}\right) = c\pi$.

Since $c\pi$ must equal 1, we must have $c = 1/\pi$ so that f is a probability density function.

(b) $\displaystyle P(-1 < X < 1) = \int_{-1}^1 \frac{1/\pi}{1+x^2}\,dx = \frac{2}{\pi}\int_0^1 \frac{1}{1+x^2}\,dx = \frac{2}{\pi}\left[\tan^{-1}x\right]_0^1 = \frac{2}{\pi}\left(\frac{\pi}{4} - 0\right) = \frac{1}{2}$

6. (a) For $0 \le x \le 1$, we have $f(x) = kx^2(1-x)$, which is nonnegative if and only if $k \ge 0$. Also,

$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 kx^2(1-x)\,dx = k\int_0^1 (x^2 - x^3)\,dx = k\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = k/12$. Now $k/12 = 1 \quad\Leftrightarrow\quad k = 12$.

Therefore, f is a probability density function if and only if $k = 12$.

(b) Let $k = 12$.

$$P\left(X \ge \tfrac{1}{2}\right) = \int_{1/2}^{\infty} f(x)\,dx = \int_{1/2}^{1} 12x^2(1-x)\,dx = \int_{1/2}^{1}(12x^2 - 12x^3)\,dx = \left[4x^3 - 3x^4\right]_{1/2}^{1}$$
$$= (4-3) - \left(\tfrac{1}{2} - \tfrac{3}{16}\right) = 1 - \tfrac{5}{16} = \tfrac{11}{16}$$

(c) The mean

$\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^1 x \cdot 12x^2(1-x)\,dx = 12\int_0^1 (x^3 - x^4)\,dx = 12\left[\frac{1}{4}x^4 - \frac{1}{5}x^5\right]_0^1 = 12\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{12}{20} = \frac{3}{5}$.

7. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, (1) $f(x) \ge 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Since $f(x) = 0$ or $f(x) = 0.1$, condition (1) is satisfied. For condition (2), we see that $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} 0.1\,dx = \left[\frac{1}{10}x\right]_0^{10} = 1$. Thus, $f(x)$ is a probability density function for the spinner's values.

(b) Since all the numbers between 0 and 10 are equally likely to be selected, we expect the mean to be halfway between the endpoints of the interval; that is, $x = 5$.

$$\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{10} x(0.1)\,dx = \left[\tfrac{1}{20}x^2\right]_0^{10} = \tfrac{100}{20} = 5, \quad \text{as expected.}$$

8. (a) As in the preceding exercise, (1) $f(x) \ge 0$ and (2) $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} f(x)\,dx = \frac{1}{2}(10)(0.2)$ [area of a triangle] $= 1$.

So $f(x)$ is a probability density function.

(b) (i) $P(X < 3) = \int_0^3 f(x)\,dx = \frac{1}{2}(3)(0.1) = \frac{3}{20} = 0.15$

(ii) We first compute $P(X > 8)$ and then subtract that value and our answer in (i) from 1 (the total probability).
$P(X > 8) = \int_8^{10} f(x)\,dx = \frac{1}{2}(2)(0.1) = \frac{2}{20} = 0.10$. So $P(3 \le X \le 8) = 1 - 0.15 - 0.10 = 0.75$.

(c) We find equations of the lines from $(0,0)$ to $(6, 0.2)$ and from $(6, 0.2)$ to $(10, 0)$, and find that

$$f(x) = \begin{cases} \frac{1}{30}x & \text{if } 0 \le x < 6 \\ -\frac{1}{20}x + \frac{1}{2} & \text{if } 6 \le x < 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^6 x\left(\tfrac{1}{30}x\right)dx + \int_6^{10} x\left(-\tfrac{1}{20}x + \tfrac{1}{2}\right)dx = \left[\tfrac{1}{90}x^3\right]_0^6 + \left[-\tfrac{1}{60}x^3 + \tfrac{1}{4}x^2\right]_6^{10}$$
$$= \tfrac{216}{90} + \left(-\tfrac{1000}{60} + \tfrac{100}{4}\right) - \left(-\tfrac{216}{60} + \tfrac{36}{4}\right) = \tfrac{16}{3} = 5.\overline{3}$$

9. We need to find m so that $\int_m^{\infty} f(t)\,dt = \frac{1}{2} \quad\Rightarrow\quad \lim\limits_{x\to\infty}\int_m^x \frac{1}{5}e^{-t/5}\,dt = \frac{1}{2} \quad\Rightarrow\quad \lim\limits_{x\to\infty}\left[\frac{1}{5}(-5)e^{-t/5}\right]_m^x = \frac{1}{2} \quad\Rightarrow$
$(-1)(0 - e^{-m/5}) = \frac{1}{2} \quad\Rightarrow\quad e^{-m/5} = \frac{1}{2} \quad\Rightarrow\quad -m/5 = \ln\frac{1}{2} \quad\Rightarrow\quad m = -5\ln\frac{1}{2} = 5\ln 2 \approx 3.47$ min.

10. (a) $\mu = 1000 \quad\Rightarrow\quad f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \ge 0 \end{cases}$

(i) $P(0 \le X \le 200) = \int_0^{200} \frac{1}{1000}e^{-t/1000}\,dt = \left[-e^{-t/1000}\right]_0^{200} = -e^{-1/5} + 1 \approx 0.181$

(ii) $P(X > 800) = \int_{800}^{\infty} \frac{1}{1000} e^{-t/1000}\, dt = \lim\limits_{x\to\infty} \left[-e^{-t/1000}\right]_{800}^{x} = 0 + e^{-4/5} \approx 0.449$

(b) We need to find m so that $\int_m^\infty f(t)\, dt = \frac{1}{2} \Rightarrow \lim\limits_{x\to\infty} \int_m^x \frac{1}{1000} e^{-t/1000}\, dt = \frac{1}{2} \Rightarrow \lim\limits_{x\to\infty} \left[-e^{-t/1000}\right]_m^x = \frac{1}{2} \Rightarrow$

$0 + e^{-m/1000} = \frac{1}{2} \Rightarrow -m/1000 = \ln\frac{1}{2} \Rightarrow m = -1000 \ln\frac{1}{2} = 1000 \ln 2 \approx 693.1$ h.

11. We use an exponential density function with $\mu = 2.5$ min.

(a) $P(X > 4) = \int_4^\infty f(t)\, dt = \lim\limits_{x\to\infty} \int_4^x \frac{1}{2.5} e^{-t/2.5}\, dt = \lim\limits_{x\to\infty} \left[-e^{-t/2.5}\right]_4^x = 0 + e^{-4/2.5} \approx 0.202$

(b) $P(0 \le X \le 2) = \int_0^2 f(t)\, dt = \left[-e^{-t/2.5}\right]_0^2 = -e^{-2/2.5} + 1 \approx 0.551$

(c) We need to find a value a so that $P(X \ge a) = 0.02$, or, equivalently, $P(0 \le X \le a) = 0.98 \Leftrightarrow$

$\int_0^a f(t)\, dt = 0.98 \Leftrightarrow \left[-e^{-t/2.5}\right]_0^a = 0.98 \Leftrightarrow -e^{-a/2.5} + 1 = 0.98 \Leftrightarrow e^{-a/2.5} = 0.02 \Leftrightarrow$

$-a/2.5 = \ln 0.02 \Leftrightarrow a = -2.5 \ln\frac{1}{50} = 2.5 \ln 50 \approx 9.78$ min ≈ 10 min. The ad should say that if you aren't served within 10 minutes, you get a free hamburger.

12. (a) With $\mu = 69$ and $\sigma = 2.8$, we have $P(65 \le X \le 73) = \displaystyle\int_{65}^{73} \frac{1}{2.8\sqrt{2\pi}} \exp\left(-\frac{(x-69)^2}{2 \cdot 2.8^2}\right) dx \approx 0.847$

(using a calculator or computer to estimate the integral).

(b) $P(X > 6 \text{ feet}) = P(X > 72 \text{ inches}) = 1 - P(0 \le X \le 72) \approx 1 - 0.858 = 0.142$, so 14.2% of the adult male population is more than 6 feet tall.

13. $P(X \ge 10) = \displaystyle\int_{10}^{\infty} \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x-9.4)^2}{2 \cdot 4.2^2}\right) dx$. To avoid the improper integral we approximate it by the integral from 10 to 100. Thus, $P(X \ge 10) \approx \displaystyle\int_{10}^{100} \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x-9.4)^2}{2 \cdot 4.2^2}\right) dx \approx 0.443$ (using a calculator or computer to estimate the integral), so about 44 percent of the households throw out at least 10 lb of paper a week.

Note: We can't evaluate $1 - P(0 \le X \le 10)$ for this problem since a significant amount of area lies to the left of $X = 0$.

14. (a) $P(0 \le X \le 480) = \displaystyle\int_0^{480} \frac{1}{12\sqrt{2\pi}} \exp\left(-\frac{(x-500)^2}{2 \cdot 12^2}\right) dx \approx 0.0478$ (using a calculator or computer to estimate the integral), so there is about a 4.78% chance that a particular box contains less than 480 g of cereal.

(b) We need to find μ so that $P(0 \le X < 500) = 0.05$. Using our calculator or computer to find $P(0 \le X \le 500)$ for various values of μ, we find that if $\mu = 519.73$, $P = 0.05007$; and if $\mu = 519.74$, $P = 0.04998$. So a good target weight is at least 519.74 g.

15. (a) $P(0 \le X \le 100) = \displaystyle\int_0^{100} \frac{1}{8\sqrt{2\pi}} \exp\left(-\frac{(x-112)^2}{2 \cdot 8^2}\right) dx \approx 0.0668$ (using a calculator or computer to estimate the integral), so there is about a 6.68% chance that a randomly chosen vehicle is traveling at a legal speed.

(b) $P(X \ge 125) = \displaystyle\int_{125}^{\infty} \frac{1}{8\sqrt{2\pi}} \exp\left(-\frac{(x-112)^2}{2 \cdot 8^2}\right) dx = \int_{125}^{\infty} f(x)\, dx$. In this case, we could use a calculator or computer to estimate either $\int_{125}^{300} f(x)\, dx$ or $1 - \int_0^{125} f(x)\, dx$. Both are approximately 0.0521, so about 5.21% of the motorists are targeted.

16. $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \quad\Rightarrow\quad f'(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \dfrac{-2(x-\mu)}{2\sigma^2} = \dfrac{-1}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}(x-\mu) \quad\Rightarrow$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}}\left[e^{-(x-\mu)^2/(2\sigma^2)} \cdot 1 + (x-\mu)e^{-(x-\mu)^2/(2\sigma^2)}\, \frac{-2(x-\mu)}{2\sigma^2}\right]$$
$$= \frac{-1}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right] = \frac{1}{\sigma^5\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\left[(x-\mu)^2 - \sigma^2\right]$$

$f''(x) < 0 \quad\Rightarrow\quad (x-\mu)^2 - \sigma^2 < 0 \quad\Rightarrow\quad |x-\mu| < \sigma \quad\Rightarrow\quad -\sigma < x-\mu < \sigma \quad\Rightarrow\quad \mu - \sigma < x < \mu + \sigma$ and similarly, $f''(x) > 0 \quad\Rightarrow\quad x < \mu - \sigma$ or $x > \mu + \sigma$. Thus, f changes concavity and has inflection points at $x = \mu \pm \sigma$.

17. $P(\mu - 2\sigma \le X \le \mu + 2\sigma) = \displaystyle\int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp\!\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$. Substituting $t = \dfrac{x-\mu}{\sigma}$ and $dt = \dfrac{1}{\sigma}\,dx$ gives us $\displaystyle\int_{-2}^{2} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2}(\sigma\,dt) = \frac{1}{\sqrt{2\pi}}\int_{-2}^{2} e^{-t^2/2}\,dt \approx 0.9545.$

18. Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ce^{-cx} & \text{if } x \ge 0 \end{cases}$ where $c = 1/\mu$. By using parts, tables, or a CAS, we find that

(1): $\int xe^{bx}\,dx = (e^{bx}/b^2)(bx-1)$

(2): $\int x^2e^{bx}\,dx = (e^{bx}/b^3)(b^2x^2 - 2bx + 2)$

Now
$$\sigma^2 = \textstyle\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx = \int_{-\infty}^{0}(x-\mu)^2 f(x)\,dx + \int_0^{\infty}(x-\mu)^2 f(x)\,dx$$
$$= 0 + \lim_{t\to\infty} c\textstyle\int_0^t (x-\mu)^2 e^{-cx}\,dx = c \cdot \displaystyle\lim_{t\to\infty} \textstyle\int_0^t \left(x^2e^{-cx} - 2x\mu e^{-cx} + \mu^2 e^{-cx}\right)dx$$

Next we use (2) and (1) with $b = -c$ to get

$$\sigma^2 = c\lim_{t\to\infty}\left[-\frac{e^{-cx}}{c^3}\left(c^2x^2 + 2cx + 2\right) - 2\mu\frac{e^{-cx}}{c^2}(-cx-1) + \mu^2\frac{e^{-cx}}{-c}\right]_0^t$$

Using l'Hospital's Rule several times, along with the fact that $\mu = 1/c$, we get

$$\sigma^2 = c\left[0 - \left(-\frac{2}{c^3} + \frac{2}{c}\cdot\frac{1}{c^2} + \frac{1}{c^2}\cdot\frac{1}{-c}\right)\right] = c\left(\frac{1}{c^3}\right) = \frac{1}{c^2} \quad\Rightarrow\quad \sigma = \frac{1}{c} = \mu$$

6 Review

CONCEPT CHECK

1. (a) See Section 6.1, Figure 2 and Equations 6.1.1 and 6.1.2.

(b) Instead of using "top minus bottom" and integrating from left to right, we use "right minus left" and integrate from bottom to top. See Figures 9 and 10 in Section 6.1.

2. The numerical value of the area represents the number of meters by which Sue is ahead of Kathy after 1 minute.

3. (a) See the discussion in Section 6.2, near Figures 2 and 3, ending in the Definition of Volume.

(b) See the discussion between Examples 5 and 6 in Section 6.2. If the cross-section is a disk, find the radius in terms of x or y and use $A = \pi(\text{radius})^2$. If the cross-section is a washer, find the inner radius r_{in} and outer radius r_{out} and use $A = \pi\left(r_{\text{out}}^2\right) - \pi\left(r_{\text{in}}^2\right)$.

4. (a) $V = 2\pi rh\,\Delta r = (\text{circumference})(\text{height})(\text{thickness})$